UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS FÍSICAS

Departamento de Física Teórica II

TESIS DOCTORAL

Discrete integrable systems, matrix orthogonal polynomials and Riemann-Hilbert problems
Sistemas integrables discretos, polinomios matriciales ortogonales y problemas de Riemann-Hilbert

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Giovanni Cassatella Contra

Directores

Manuel Mañas Baena
Piergiulio Tempesta

Madrid, 2016

DEPARTAMENTO DE FÍSICA TEÓRICA II
FACULTAD DE CIENCIAS FÍSICAS

UNIVERSIDAD
COMPLUTENSE
MADRID

Discrete Integrable Systems, Matrix Orthogonal Polynomials and Riemann-Hilbert Problems

Sistemas Integrables Discretos, Polinomios Matriciales Ortogonales y Problemas de Riemann-Hilbert

PhD Thesis

CANDIDATE:
GIOVANNI CASSATELLA CONTRA

SUPERVISORS:
MANUEL MAÑAS BAENA
PIERGIULIO TEMPESTA

Madrid
2015

Contents

AGRADECIMIENTOS

Hubiese sido muy complicado realizar este trabajo sin la ayuda y el apoyo de algunas personas.

En primer lugar quería agradecer a mis directores de tesis Manuel Mañas y Piergiulio Tempesta. A parte de por haberme aceptado como estudiante, por la paciencia y la dedicación que han tenido conmigo, y por haberme introducido al mundo de la investigación. También quería agradecer especialmente a Gabriel Álvarez, quien me ha ayudado a utilizar el programa Mathematica para encontrar resultados numéricos de las ecuaciones que hemos desarrollado, a Decio Levi, de la Universidad de Roma III, a David Fernández y Ángela Alera.

También quería agradecer a mi padre, a mi madre y a mi hermano, y a otras personas que me han brindado un apoyo fundamental.

Tampoco se me olvida agradecer a los que han sido mis compañeros de despacho y otros compañeros, con los que he pasado buenos momentos dentro y fuera de la facultad.

Summary

The purpose of this Thesis is to relate the notion of *integrability* for discrete systems with the theory of matrix orthogonal polynomials, by using a Riemann–Hilbert approach.

The study of integrable models has originated in Classical Mechanics, in relation with the problem of solving Newton's equations of motion [**2**]. The work of Liouville, Hamilton, Jacobi and others firmly established integrable systems as prototype modes "solvable by quadratures", i.e. by a direct integration procedure [**7**]. An impressive amount of research has been devoted to the study of the geometry of classical integrable and superintegrable systems [**66**], [**82**], especially in relation with the problem of separation of variables for the associated Hamilton-Jacobi equation [**75**].

In the second half of twentieth century, the discovery of the Inverse Scattering Method for the Korteweg-de Vries equation [**42, 43**] signed the beginning of a new field of research: the study of integrable systems with infinitely many degrees of freedom, expressed in terms of nonlinear field equations. New classes of integrable models, often encompassed in hierarchies of nonlinear partial differential equations, were introduced. In particular, equations possessing soliton solutions found interesting applications, for instance, in classical hydrodynamics and quantum optics.

In the last three decades, in the community of researchers on integrable systems there has been a growing interest in the study of *discrete models*, i.e. dynamical systems defined in a lattice of points, and represented in terms of difference equations.

Many analytic techniques used for the treatment of continuous equations were soon extended to the discrete world. The motivation for the study of discrete integrable equations relies at least on two aspects.

First, in many circumstances the natural phenomena are more conveniently represented in terms of discontinuous temporal steps. This is the

case, for instance, when dealing with models from evolutive biology, economy, decision processes, neural networks, etc. In these cases, recurrence equations or functional relations naturally arise.

Second, from a mathematical point of view, discrete equations seem more fundamental than their continuous counterparts. Needless to say, these objects possess many fascinating properties, making their study particularly exciting.

In the present work, several analytic and algebraic techniques, originally developed in the study of continuous equations, are used to investigate the properties of a new class of integrable systems: *matrix discrete models*. These models, usually described in terms of matrix recursion equations, represent a recent acquisition in the realm of integrability.

The models studied in this Thesis can be considered to be a matrix version of the discrete Painlevé equations [**72, 73**]. The classical continuous Painlevé equations were introduced at the beginning of twentieth century to classify equations with movable singularities, and play a major role in the modern Mathematical Physics [**24**], [**25**]. Recently, applications of these equations were found also in 2D quantum gravity [**38**] and topological field theories [**31**].

Soon, multiple connections emerged, at a fundamental level, between the so called Painlevé property and the concept of integrability. This connection motivated the introduction of an algebraic, discrete analogue of the Painlevé property, called the *singularity confinement* [**46**]. The key observation behind this notion is the fact that, for integrable discrete models, it turns out that a singularity appearing in the lattice of independent variables disappears after making evolve the system via a finite number of iteration steps.

In this work, new matrix integrable models, generalizing some of discrete Painlevé equations known in the literature, are constructed. They are obtained by applying an important analytic technique, the Riemann–Hilbert approach [**50**], which was developed for solving a great variety of problems in pure and applied mathematics. In simple terms, the Riemann–Hilbert problem aims at reconstructing an analytic function from certain jump conditions in the complex plane, or equivalently to the analytic factorization of a given scalar or matrix-valued function on a curve.

In 1992, in [**39**] the Riemann–Hilbert method was related to the theory of orthogonal polynomials. This is another fundamental ingredient for our research. Indeed, we shall derive our integrable matrix models by using a generalization of the Riemann–Hilbert technique, where now orthogonal polynomials with matrix coefficients are used instead of scalar ones. Precisely, we have focused on an interesting class of polynomials, named Freud polynomials [**40**], in the two cases when they are defined on the real line and on the circle, respectively.

The main results of this work can be summarized as follows.

1) We have generalized and solved the Riemann–Hilbert problem to the case of matrix orthogonal polynomials [**18**]. We considered the case of matrix Freud polynomials, both on the real line and on the unit circle.

2) Novel matrix integrable models have been derived [**20**]. They can be considered to be generalizations of the discrete Painlevé equations I and II, which in their scalar versions have been proposed by Van Assche in [**8**].

3) A thorough analysis of the singularity confinement properties of the matrix discrete Painlevé equation I obtained by solving the Riemann–Hilbert problem on the line has been performed [**19**]. We have shown analytically that the property holds generically, i.e. for a large set of initial conditions. The set of conditions where it does not hold is represented by specific algebraic varieties in the space of parameters.

In short, our analysis paves the way to a formulation of a theory of matrix integrable discrete models, still largely unexplored, based on a Riemann–Hilbert approach. Interestingly enough, the standard approaches used in soliton theory, in particular the discrete version of the Painlevé singularity analysis, keep playing a crucial role in the matrix theory we wish to develop. Our research represent a first exploration of this new exciting field.

The Thesis is organized in four Chapters and an Appendix. They correspond to three published articles and a preprint.

In Chapter 1, a summary of the state of art of the current research in the topics of the Thesis is presented. After an historical excursus on the notion of integrability, the main mathematical techniques used in the subsequent considerations are presented.

Chapter 2 contains the article "Riemann–Hilbert Problems, Matrix Orthogonal Polynomials and Discrete Matrix Equations with Singularity Confinement", by G. Cassatella and M. Mañas, Studies in Applied Mathematics, 128, 252-274, 2011. We present the solution of the Riemann–Hilbert problem for matrix orthogonal polynomials of Freud-type, corresponding to a quartic potential and defined on the real line. The matrix recursion relations emerging from the solution of the Riemann–Hilbert problem allow to define a matrix version of the discrete Painlevé equation I. Its singularity confinement properties are preliminary studied in the case of triangular initial conditions.

Chapter 3 contains the article "Singularity confinement for matrix discrete Painlevé equations, by G. Cassatella Contra, M. Mañas and P. Tempesta, Nonlinearity 27, 2321-2335, 2014. In this paper, the matrix discrete equation obtained in Chapter 2 is studied analytically. In particular, the singularity confinement is proven to hold generically. In other words, our matrix system is confined for an arbitrary choice of the initial conditions, except for a set of algebraic varieties in the space of parameters.

Chapter 4 is based on the paper "Freud polynomials on the circle and a matrix Painlevé II discrete equation", by G. Cassatella Contra, M. Mañas and P. Tempesta, preprint 2015. Here the analysis performed in Chapter 2 is extended to the case of a matrix Riemann–Hilbert problem associated with matrix orthogonal polynomials on the unit cirle. The problem is solved in full generality. For a class of matrix Freud polynomials, a matrix discrete version of the Painlevé equation II is derived. The singularity properties of this matrix models are analyzed in some particular cases, where it is shown that the singularity confinement property holds.

The final Appendix contains the article "Discrete Multiscale Analysis: a Biatomic Lattice System", by G. Cassatella Contra and D. Levi. The paper discusses the multiscale reductive perturbative approach for discrete systems. In particular, a new version of a discrete nonlinear Schrödinger equation is obtained from a multiscale analysis of the discrete equations of motion of a biatomic lattice system.

Resumen del trabajo de Tesis

El propósito de esta tesis doctoral es el estudio de la conexión, mediante el problema de Riemann–Hilbert, entre sistemas discretos y la teoría de polinomios matriciales ortogonales.

La investigación de los modelos integrables se originó en la Mecánica Clásica, en relación a la resolución de las ecuaciones de Newton [**2**]. Los trabajos de Liouville, Hamilton, Jacobi y otros sentaron las bases de los sistemas integrables como prototipos modelos resolubles por cuadraturas, v.g., por integración directa [**7**]. Hay una cantidad importante de investigación dedicada a los aspectos geométricos de los sistemas clasicos integrables y superintegrables [**66**], [**82**], especialmente en relación a la separación de variables de la ecuación de Hamilton–Jacobi [**75**].

Fue la aplicación, en la segunda mitad del siglo pasado, de la transformada espectral inversa para la resolución del problema de Cauchy de la ecuación de Korteweg–de Vries [**42, 43**] la que marco el inicio de una nueva etapa en este campo, el del estudio de sistemas integrables con un número infinito de grados de libertad, que generalmente se expresan en términos de jerarquías de ecuaciones no lineales en derivadas parciales. Particularmente reseñable, por su aplicación en la hidrodinámica y en la óptica cuántica, es la aparición de las soluciones a un número de solitones arbitrario.

En las últimas tres décadas ha habido un importante interés por el estudio de modelos discretos, v.g., sistemas dinámicos definidos en un retículo de puntos, y expresados en términos de ecuaciones no lineales en diferencia parciales. Muchas de las técnicas encontradas en el mundo continuo se extendieron a este nuevo contexto discreto. Hay dos razones fundamentales para este interés.

En primer lugar, es necesario señalar que en muchas circunstancias los fenómenos naturales se describen mejor en términos de un flujo temporal discreto. Este es el caos, por ejemplo, de modelos en biología evolutiva, economía, procesos de decisión y redes neuronales. En estos ejemplos las relaciones de recurrencia o reacciones funcionales, aparecen naturalmente.

En segundo lugar, y desde un punto de vista matemático, las ecuaciones discretas pueden ser en ciertos contextos más fundamentales que sus versiones continuas. Subrayemos también las muchas e importantes propiedades que poseen éstas.

En esta tesis, utilizamos diversas técnicas analíticas y algebraicas, diseñadas originalmente para el estudio del caso continuo, son usadas para investigar las propiedades de una nueva clase de sistemas integrables: *modelos discretos matricales*. Estos modelos son descritos habitualmente en términos de ecuaciones de recurrencia matriciales, representan un reciente avance en la teoría de los sistemas integrables.

Los modelos que estudiamos se pueden considerar como versiones matriciales de ecuaciones de Painlevé discretas [**72, 73**]. Las ecuaciones de Painlevé continuas se introdujeron al comienzo del siglo pasado cuando se intentaba clasificar singularidades móviles, y juega un papel predominante en la Física matemática moderna [**24**], [**25**]. Recientemente se han encontrado aplicaciones de las mismas en gravedad cuántica bidimensional [**38**] y en teorías topológicas de campos [**31**].

Enseguida se encontraron múltiples conexiones fundamentales entre la propiedad de Painlevé y el concepto de integrabilidad. Esta conexión motivó la introducción de un análogo algebraico y discreto de la propiedad de Painlevé, que fue llamada *confinamiento de singularidades* [**46**]. La idea clave tras está propiedad es simple, las singularidades pueden aparecer, pero tras una pocas interacciones desaparecerán.

Pues bien, hemos sido capaces de construir en esta tesis nuevos sistemas integrables que extienden algunas ecuaciones de Painlevé discretas conocidas en la literatura. Las obtenemos mediante una importante herramienta analítica, el método de Riemann–Hilbert [**50**], que permite la resolución de un número muy amplio de problemas tanto en matemática pura como en matemática aplicada. En términos sencillos, el problema de Riemann–Hilbert consiste en reconstruir una función analítica conocidos sus saltos o discontinuidades en el plano complejo o, equivalentemente, la factorización analítica de una función escalar o matricial en una curva.

En 1992 [**39**] el método de Riemann–Hilbert se relaciono con los polinomios ortogonales, lo que constituye otro de los pilares de nuestra investigación. Así, nosotros derivaremos nuestros sistemas integrables usando una generalización de la técnica de Riemann–Hilbert, donde los polinomios ortogonales escalares son reemplazados por sus versiones matriciales. Para ser más precisos, nos centraremos en los polinomios de Freud [**40**], tanto en la recta real como en el círculo unitario.

Los resultados principales de la tesis se pueden resumir en:

(1) Hemos generalizado y resuelto el problema de Riemann–Hilbert para los polinomios matriciales ortogonales [**18**]. Donde consideramos polinomios de Freud tanto en la recta real como en el círculo unitario.
(2) Se han derivado nuevos sistemas integrables [**20**]. Estos se pueden considerar como extensiones matriciales de las ecuaciones de Painlevé discretas I y II. En el caso escalar fueron encontradas por Van Assche [**8**].
(3) Hemos realizado un análisis completo del confinamiento de singularidades la la ecuación matricial discreta de Painlevé I obtenida en la resolución del problema de Riemann–Hilbert en la recta real [**19**]. Hemos demostrado analíticamente la presencia de dicha propiedad genéricamente. El conjunto de condiciones iniciales en donde no se da definen variedades algebraicas de codimension superior o igual a uno en el espacio de parámetros.

Esta tesis se organiza en cuatro capítulos y un apéndice. Corresponden a tres artículos publicados y a un *preprint*.

Capítulo 1. Aquí realizamos un resumen del estado del arte hoy en día de los diferentes temas tratados en esta tesis. Tras una excursión a la noción de integrabilidad, presentamos las principales técnicas matemáticas lrequeridas por desarrollos posteriores.

Capítulo 2. Contiene el artículo *Riemann–Hilbert Problems, Matrix Orthogonal Polynomials and Discrete Matrix Equations with Singularity Confinement*, por G. A. Cassatella-Contra y M. Mañas, publicado en Studies in Applied Mathematics, **128** (2011) 252-274. Aquí se presenta la solución a un problema de Riemann–Hilbert para polinomios matriciales ortogonales en la recta real de tipo Freud, correspondientes a un potencial cuártico. Las relaciones de recurrencia matriciales, que emergen del problema de Riemann–Hilbert, permiten encontrar una ecuación matricial discreta de Painlevé I. Las propiedades de confinamiento de singularidades se encuentran para el caso en que las condiciones iniciales son matrices triangulares.

Capítulo 3. Contiene el artículo *Singularity confinement for matrix discrete Painlevé equations*, por G. A. Cassatella-Contra, M. Mañas y P.

Tempesta publicado en Nonlinearity **27** (2014)2321-2335. En este articulo la ecuación matricial discreta de Painlevé I presentada en el Capítulo 2 es estudiada en toda su generalidad. Se demuestra que el confinamiento de singularidades se da para cualquier condición inicial de forma genérica, esto salvo por variedades algebraicas de codimension no nula en el espacio de parámetros.

Capítulo 4. Esta basado en el preprint *Freud polynomials on the circle and a matrix Painlevé II discrete equation* de G. A. Cassatella-Contra, M. Mañas y P. Tempesta (2015). Ahora el análisis del capítulo 2 se extiende a los polinomios matriciales ortogonales sobre el el circulo unidad. El problema se resuelve en toda su generalidad, y para una clase de polinomios de Freud matriciales se obtiene la ecuación matricial discreta de Painlevé II. La existencia del confinamiento de singularidades se demuestra que es cierta en algunas situaciones.

Apéndice. Contiene el artículo *Discrete Multiscale Analysis: a Biatomic Lattice System*, por G. A Cassatella Contra y D. Levi y publicado en Journal of Nonlinear Mathematical Physics, **17** (2010) 357–377. Se discute una aproximación a sistemas discretos basada en desarrollos perturbativos multi-escala reductivos. Se presenta una ecuación de Schrödinger no lineal discreta, resultante de una análisis multiescala, para el movimiento de un retículo biatómico.

List of publications

I) G. Cassatella Contra, M. Mañas and P. Tempesta, *Freud Polynomials on the Circle and a Matrix Painlevé II Discrete Equation*, preprint 2015.

II) G. Cassatella Contra, M. Mañas and P. Tempesta, *Singularity confinement for matrix discrete Painlevé equations*, Nonlinearity **27**, 2321-2335 (2014).

III) G. Cassatella, M. Mañas, *Riemann–Hilbert Problems, Matrix Orthogonal Polynomials and Discrete Matrix Equations with Singularity Confinement*, G. Cassatella, M. Mañas, Studies in Applied Mathematics, **128**, 252-274 (2011).

IV) G. Cassatella Contra and D. Levi *Discrete Multiscale Analysis: a Biatomic Lattice System*, J. Nonlinear Math. Phys. **17**, 137-177 (2010).

Work in progress

V) G. Cassatella Contra, M. Mañas and P. Tempesta, *Non-Hermitian matrix models and Matrix Freud Orthogonal Polynomials on the Circle*, 2015.

CHAPTER 1

Introduction: An excursus on integrability, matrix orthogonal polynomials, Riemann–Hilbert problems, singularity analysis.

1. Historical perspective

Integrable models are widely recognized as paradigmatic examples of systems possessing mathematical beauty and physical relevance. Their study therefore represents an important research area of Theoretical and Mathematical Physics, still particularly active.

The theory of integrable systems originated in Classical Mechanics, in relation with the problem of solving Newton's equations of motion. Apart very few examples solved analytically, the first substantial progress was made by Liouville, who was able to propose the first theory of integrability, based on the notion of solvability "by quadratures". This amounts to say that there exists a maximal set of Poisson commuting invariants. Other foundational contributions came from the work by Euler, Lagrange, Jacobi, Hamilton, etc.

Integrability was also related to the problem of finding separating variables for the Hamilton-Jacobi equation, perhaps the most important problem of classical mechanics. In the last decades, several new topological and geometrical approaches, due to Levi-Civita, Arnold, Magri, Sklyanin (among others) have been proposed to solve this problem for the case of finite-dimensional systems, including bi-Hamiltonian and quasi-bi-Hamiltonian approaches, the theory of Nijenhuis and Haantjes tensors, etc.

At the same time, during the XX century, the theory of integrable systems grew progressively towards another fundamental direction: the study of infinite dimensional models, i.e. integrable nonlinear partial differential equations. This class of equations was recognized of outmost importance since the discovery of soliton solutions, made empirically by S. Russell in

1834. In 1895, Russel's experiments were interpreted by Dutch physicist Diederick Korteweg and his student Gustav de Vries in terms of a nonlinear partial differential equation [**55**]. This equation, nowadays called Korteweg–de Vries equation (KdV), was already known to Boussinesq, see footnote on page 360 of [**14**]. It reads

$$u_t + 6uu_x + u_{xxx} = 0, \qquad u = u(x,t).$$

Nevertheless, this crucial work was ignored for decades by mathematicians, physicists, and engineers studying water waves. Only 1965 with Zabusky and Kruskal work [**84**] there was a resurgence of interest in nonlinear field equations as the KdV equation.

Soon after, the study of infinite-dimensional models experienced an astonishing development since the discovery of the Inverse Scattering Transform Method, due to Gardner, Green, Kruskal and Miura [**42, 43**], Lax [**60**], Zakharov and Shabat [**85, 86**]. This revolutionary technique allowed the solution of the Korteweg-de Vries equation with initial data rapidly decreasing at infinity.

Another central idea was that of Lax pairs [**60**] which consists in associating to a nonlinear PDE two linear operators (Lax pair) L and M. One of them is an eigenvalue problem for eigenvalues taken to be independent of time; the other one is responsible of the time evolution. The compatibility condition between these two operators is equivalent to the original PDE.

Starting from these seminal ideas, the theory of integrable systems expanded considerably to encompass new areas as exactly solvable quantum field models, hierarchies of nonlinear PDEs, ordinary differential equations of Painlevé type, Also, a new class of quantum and classical integrable systems à la Liouville were discovered. In the last two decades, new geometric structures, as Frobenius structures and bi-Hamiltonian systems, were introduced as fundamental objects providing a unifying framework in the discussion of integrability.

At the same time, in the integrable systems community, there was a growing interest in a different, but related class of systems: the *discrete* ones.

A discrete model is a system described by a set of difference equations, i.e. equations whose independent variables are defined on a lattice of points, usually taken to be regular (i.e. with a pre-determined geometry). This kind of systems is of utter importance in many contexts, as evolutive biology, economics, social sciences, etc. In quantum gravity, the existence of a fundamental length (called the Planck length), makes the theory as an intrinsically discrete one. Physically relevant discrete integrable models have been considered in the context of field theories and Hamiltonian gravity, for instance in [**41**], [**49**], [**74**]. Generally speaking, difference equations are a fundamental piece of the rapidly expanding area of discrete mathematics.

The study of difference equations dates back to the work of Newton, and developed in parallel to that of differential equations. Many algebraic and analytic techniques are nowadays available to find their exact solutions. In particular, symmetry methods are particularly relevant. The case of linear equations, widely investigated lead to a body of results which is up to some extent comparable to that of differential equations.

Much more challenging are the nonlinear discrete systems, which are one of the main object of interest of the present Thesis. As is well known, nonlinear maps can lead to a chaotic behaviour, as is the case of the logistic map or the standard map. Nevertheless, there are systems which are still integrable, in the sense that they possess many of the classical regularity and deterministic properties shared by the continuous integrable models.

One of the most paradigmatic discrete integrable models is the Toda lattice, proposed by M. Toda [**83**]. It describes the evolution of a nonlinear chain of particles, connected by means of a nonlinear interaction expressed by an exponential potential of the form $\phi(r) = a/be^{-br} + ar + const$. Needless to say, this research line was motivated by the famous experiment that E. Fermi, J. Pasta and S. Ulam performed on a model of coupled nonlinear oscillators, nowadays called the Fermi-Pasta-Ulam (FPU) model. Toda was able to obtain exact solutions of his model in terms of elliptic functions. They represent a sort of generalization of the standard normal oscillation modes of linear chains. Soon, the integrabliliy à la Liouville was established by Henon. Also, Flashka was able to introduce a system of variables that allowed the Lax formulation of Toda system.

Once the discrete integrability was recognized to be a notion as relevant as the classical one (even, perhaps, a more fundamental one), the usual analytic techniques developed in the continuous case were rapidly extended and generalized to the discrete word.

In particular, Ablowitz and Ladik [**1**] first introduced a hierarchy of nonlinear equations, expressed in terms of an infinite set of commuting flows

[**1**]. Other integrable discrete hierarchies were discovered (among them the generalized extended Toda hierarchies), and their algebro-geometric properties were intensively investigated, in terms of Frobenius structures, bi-Hamiltonian structures, ωN manifolds, etc.

One should also notice that the problem of integrability preserving discretizations of PDEs has become a very active research area [**81**], and has been widely investigated with both geometrical and algebraic methods [**11**], [**12**], [**69**], [**53**], [**62**], [**61**]. Frobenius manifolds are also relevant in the discussion of generalized Toda systems [**17**].

The main purpose of the present Thesis is to provide a *matrix version* of discrete integrability. We will show that, indeed, a natural generalization of the well-established theory of scalar and vector discrete models can be worked out in a matrix context. Precisely, we shall study recurrences where the dependent variable is a $N \times N$ matrix function. By way of an example, we mention here one of the models which are central in our analysis. It is the following matrix discrete equation:

$$\beta_{n+1} = n\beta_n^{-1} - \beta_{n-1} - \beta_n - \alpha, \qquad n = 1, 2, \dots \tag{1}$$

where $\beta_n \in \mathbb{C}^{N\times N}$ is a $N \times N$ complex matrix. This equation naturally emerges as a matrix discrete version of the classical Painlevé I equation.

The extension of the notion of integrability to this class of models entails the generalization to the matrix case of many notions which are standard in soliton theory. In the subsequent considerations, we shall review the main ideas underlying several mathematical techniques which are crucial for the present work. Precisely, we shall focus on matrix orthogonal polynomials, Riemann–Hilbert problem and on the singularity confinement property.

2. Orthogonal Polynomials

The theory of orthogonal polynomials originated in the work by Adrien-Marie Legendre, who was interested in solving the equations of motion of celestial mechanics. The first theoretical formulations date back to the seminal work of Stieltjes [**76, 77**] and Chebychev [**21, 22**]. Due to the ubiquity nature of orthogonal polynomials in modern science, their study has been performed from very many different points of view. Indeed, they can be considered from a purely algebraic point of view, from that of approximation theory, and in the context of modern measure theory and functional analysis.

2.1. Scalar orthogonal polynomials. Let us discuss some basic properties of the theory of orthogonal polynomials. Hereafter $\{P_n(x)\}_{n\in\mathbb{N}}$, will denote a family of monic polynomials, where $P_n(x)$ is a polynomial of degree n in a variable x, with coefficients in $\mathbb{R}$.

Let μ be a positive Borel measure μ on $\mathbb{R}$ with infinite support: it assigns a positive real number to every Borel set $X \subset I$ which is countably additive.

We assume that *moments* of the measure exist for $n = 0, 1, \ldots$

$$\mu_n := \mathcal{L}(x^n) = \int_I x^n d\mu,$$

A classical theorem states that, associated with our positive Borel measure μ (with infinite support and finite moments), there exists a unique sequence of monic polynomials $\{P_n(x)\}_{n\in\mathbb{N}}$

$$P_n(x) = x^n + \text{lower order terms}, \quad n = 0, 1, \ldots,$$

and a sequence of positive numbers $\{h_n\}_{n\in\mathbb{N}}\}$, with $h_0 = 1$, such that

$$\int_{\mathbb{R}} P_n(x)P_m(y)d\mu = h_n\delta_{n,m}.$$

When $d\mu(x) = w(x)dx$ then we talk about orthogonal polynomials with respect to the *weight function* $w(z)$.

Another crucial result is the following: a monic sequence of orthogonal polynomials satisfies a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \qquad n > 0, \tag{2}$$

with

$$P_0(x) = 1, \qquad P_1(x) = x - \alpha_0,$$

where $\alpha_n \in \mathbb{R}$, for $n \geq 0$ and $\beta_n > 0$ for $n > 0$. The converse is also true: a sequence of polynomials satisfying a three-term relation is a sequence of orthogonal polynomials (Favard's theorem [**37**]).

The moments of the measure μ form a numerical sequence playing an important role in many aspects related to orthogonality. The moment problem is an inverse problem: given an arbitrary sequence $\{\mu_n\}_{n\in\mathbb{N}}$, to find the necessary and sufficient conditions ensuring that the given sequence is a sequence of momenta for an orthogonal sequence of polynomials in an interval $I \subset \mathbb{R}$. We can distinguish the Stieltjes problem on the half line, i.e. $I = [0, \infty]$, the Hamburger problem, corresponding to $I = (-\infty, \infty)$, and the Hausdorff problem, when $I = [0, 1]$.

The most widely used orthogonal polynomials are the classical orthogonal polynomials on the real line, consisting of the Hermite polynomials, the Laguerre polynomials, the Jacobi polynomials together with their special cases the Gegenbauer polynomials, the Chebyshev polynomials, and the Legendre polynomials (see e.g. the book [**3**]).

A very interesting class of orthogonal polynomials, which will be crucial in the subsequent considerations are the *Freud orthogonal polynomials* in the real line [**40**]. They are are associated to the weight

$$w_\rho(x) = |x|^\rho e^{-|x|^m}, \qquad \rho > -1, \quad m > 0.$$

The relevance of these polynomials is their direct relation with interesting discrete models. A particularly relevant observation, made in [**8**], is that for $m = 2, 4, 6$ the recursion relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x),$$

induces to a recursion relation satisfied by the recursion coefficients a_n. In particular, for $m = 4$ Van Assche obtains for a_n the discrete Painlevé I equations, which is a difference equation whose singularities are confined. (see also [**63**] and the survey [**27**]).

One of the main objective of this Thesis will be to extend these polynomials to the matrix case, and extend the results in [**8**] to a more general setting.

2.2. Matrix orthogonal polynomials: some background. The theory of matrix orthogonal polynomials is much more recent. To the best of our knowledge, orthogonal polynomials with matrix coefficients on the real line were first considered by Krein [**56, 57**] in 1949. Apart some studies, by Berezanskii (1968) [**10**], Geronimo [**44**] (1982), the subject layed dormant for several decades.

In [**6**] (1984), a kind of matrix version of Favard's theorem was found.

More recently, there were studies that showed that matrix orthogonal polynomials (MOP) may mantain certain properties, as the Rodrigues formula [**35, 36, 26**] or a second order differential equation [**32, 34, 13**], which are typical of the scalar case.

Later on, in [**33**] matrix orthogonal polynomials were regarded as eigenfunctions of operators of the form

$$D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0.$$

Moreover, in [**13**] a new family of MOP's satisfying second order differential equations whose coefficients do not behave asymptotically as the identity matrix was found (see also [**16**]).
Orthogonal polynomials on the unit circle were introduced in the 1920's by Szego [**78, 79, 80**]. They satisfy the Szego recurrence equation, i.e.

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n \Phi_n^*(z),$$

where α_n are called the Verblunsky coefficients and are such that $|\alpha_n| < 1$. The matricial version of orthogonal polynomials on the unit circle (i.e. with matricial coefficients) were originally developed by Krein [**58**] (see also related work in [**47**]).

Then many papers appeared, contributing to extend most of the properties of orthogonal polynomials on the unit circle to the matrix case. In particular, see [**51, 52, 67, 71, 15, 4, 30, 45, 64, 27**].

2.3. Matrix orthogonal polynomials on the real line. Matrix polynomials are objects of the form

$$P_n(z) = \mathbb{I}_N z^n + \gamma_n^{(1)} z^{n-1} + \ldots + \gamma_n^{(n)} \in \mathbb{C}^{N\times N} \tag{3}$$

where $\mathbb{I}_N$ denotes the identity matrix in $\mathbb{C}^{N\times N}$ and $\gamma_n^{(i)} \in \mathbb{C}^{N\times N}$, $i = 1, \ldots, n$ are the coefficient matrices.

In the present work, we shall focus on two cases: matrix orthogonal polynomials in the real line and in the unit circle.

We wish to construct a family of matrix polyomials which are orthogonal with respect to a matrix-valued measure or matrix of measures μ on $\mathbb{R}$. This measure assigns to every Borel set X countably additive a *positive semi-definite* $N \times N$ *Hermitian matrix* $\mu(X)$. Usually, a normalization condition

$$\mu(\mathbb{R}) = \mathbb{I}_N$$

is assumed.

Let us denote by dx the Lebesgue measure in $\mathbb{R}$. We assign an $N \times N$ Hermitian matrix $V = V_{i,j}(x)$. Then we choose

$$\mathrm{d}\,\mu = \rho(x)\,\mathrm{d}\,x \qquad \text{where} \quad \rho = \exp(-V(x)).$$

Also, we shall focus on the case of even potentials:

$$V(x) = V(-x).$$

Consequently, ρ is a positive semi-definite Hermitian matrix. The moments associated with μ are

$$m_j := \int_{\mathbb{R}} x^j \rho(x)\,\mathrm{d}\,x \in \mathbb{C}^{N\times N}, \qquad j = 0,1,\dots,$$

Also, we can introduce the truncated moment matrix

$$m^{(n)} := (m_{i,j}) \in \mathbb{C}^{N\times N}$$

where $m_{i,j} = m_{i+j}$ and $0 \leq i,j \leq n-1$. Then, there exists a family $\{P_n(x)\}_{n\in\mathbb{N}}$ of monic matrix orthogonal polynomials (MOP) of the form (3), satisfying the orthogonality relations

$$\int_{\mathbb{R}} P_n(x) x^j \rho(x)\,\mathrm{d}\,x = 0, \qquad j = 0,\dots,n-1$$

One can show that the condition of being invertible for the truncated moment matrix guarantees the uniqueness of the family of MOP (3).

The Cauchy transform of $P_n(z)$ will also play an important role. It is defined by

$$Q_n(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{P_n(x)}{x-z}\rho(x)\,\mathrm{d}\,x \tag{4}$$

whenever $z \notin \operatorname{supp}(\rho(x)\,\mathrm{d}\,x)$. One can prove that the Q_n possess the following asymptotic behaviour:

$$Q_n(z) = c_n^{-1} z^{-n-1} + O(z^{-n-2}), \qquad z \to \infty$$

where the coefficients c_n are defined by

$$c_n := \left(-\frac{1}{2\pi i}\int_{\mathbb{R}} P_n(x)\rho(x) x^n\,\mathrm{d}\,x\right)^{-1}. \tag{5}$$

2.4. Matrix Szegő polynomials on the unit circle. Another important class of polynomials, which will play a major role in Chapter 3, is that of matrix orthogonal polynomials on the unit circle. Hereafter, we shall sketch some of their basic properties.

We will denote the unit circle by $\mathbb{T} := \{z \in \mathbb{C} :\mid z \mid= 1\}$. Now, we shall focus on a matrix measure μ with support in $\mathbb{T}$.

We require that it satisfies $d\mu = w(z)\frac{dz}{iz}$. Here $w(z)$ is a continuous and Hermitian $N\times N$ matrix that is defined in $\mathbb{T}$, and can be expanded analytically in an annulus around $\mathbb{T}$.

Given the weight w we will suppose that the following left and right *monic matrix Szegő polynomials* P_n^L and P_n^R exist and satisfy the following orthogonality relations:

$$\int_{\mathbb{T}} P_n^L(z)z^{-j}\,\mathrm{d}\,\mu = -i\int_{\mathbb{T}} P_n^L(z)z^{-j-1}w(z)\,\mathrm{d}\,z = 0, \quad j = 0,\ldots,n-1, \tag{6}$$

and

$$\int_{\mathbb{T}} \mathrm{d}\,\mu P_n^R(z)z^{-j} = -i\int_{\mathbb{T}} w(z)P_n^R(z)z^{-j-1}\,\mathrm{d}\,z = 0, \quad j = 0,\ldots,n-1, \tag{7}$$

respectively.

Elementary properties for the left inner product are

(1)

$$\langle P, Q\rangle_L = \langle Q, P\rangle_L^*,$$

(2) If K_1, $K_2 \in \mathbb{C}^{N\times N}$, then

$$\langle K_1P_1 + K_2P_2, Q\rangle_L = K_1\langle P_1, Q\rangle_L + K_2\langle P_2, Q\rangle_L;$$

(3) $\langle P, P\rangle_L$ is nonnegative definite;

(4) $\langle P, P\rangle_L = 0$ is and only if $P = 0$.

For a matrix polynomial P_n of degree n, we define the *reverse Szegő polynomials* as

$$\tilde{P}_n(z) := z^n[P_n(1/\overline{z})]^*.$$

The reverse left Szegő polynomials satisfy the following orthogonality relations

$$\int_{\mathbb{T}} \tilde{P}_n^L(z)z^{-j}d\mu = -i\int_{\mathbb{T}} \tilde{P}_n^L(z)z^{-j-1}w(z)dz = 0, \qquad j = 1,\ldots,n, \tag{8}$$

and similarly for the reverse right polynomials.

By analogy with the previous discussion, one can introduce the Cauchy transforms of our polynomials. We have

$$Q_n^L(z) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{P_n^L(u)w(u)}{u^n(u-z)}\,\mathrm{d}\,u$$

for the Cauchy transforms of $P_n^L(z)$, and

$$\tilde{Q^L}_n(z) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{w(u)\tilde{P^L}_n(u)}{u^{n+1}(u-z)}\,\mathrm{d}\,u$$

the reverse Cauchy transforms of P_n^L. Again $z \notin \operatorname{supp}(w(u)\,\mathrm{d}\,u)$.

One can proceed analogously for Q^R and $\tilde{Q^R}$ in terms of P_n^R and $\tilde{P_n^R}$, respectively. Notice that the following result holds:

$$\tilde{Q}_n^{L,R}(z) := -z^{-n-1}[Q_n^{L,R}(1/\overline{z})]^*. \tag{9}$$

The main reason for considering matrix orthogonal polynomials in our work is the fact that the matrix systems of interest will be obtained from suitable recursion relations, that appear by solving an associated Riemann–Hilbert problem. In the following, we shall revise some ideas concerning this class of problems, of outmost relevance in modern Mathematical Physics.

3. Riemann–Hilbert problems

The celebrated Riemann–Hilbert problems, named after Bernhard Riemann and David Hilbert, are a class of problems that arise in the study of differential equations in the complex plane.

In its original formulation, a Riemann–Hilbert problem deals with Fuchsian systems of differential equations, i.e. systems of the form

$$\frac{\mathrm{d}\,\Psi(\lambda)}{\mathrm{d}\,\lambda} = A(\lambda)\Psi(\lambda). \tag{10}$$

Here, the $N \times N$ matrix $A(\lambda)$ is a rational function of λ whose singularities are single poles. What is called the monodromy group of the system (10) is a conjugacy class of representations of the fundamental group of the Riemann sphere (punctured at the poles of $A(\lambda)$) in the group of $N \times N$ invertible matrices.

The problem of the existence of a Fuchsian system with given poles and a given monodromy group represented the problem twenty-one of the famous list of Hilbert, and was later called the "Riemann-Hilbert" problem (see [**5**] for a review of the story of this problem and the origin of the name "Riemann-Hilbert").

At the same time, both in pure and applied mathematics, a related (but for many aspects independent) analytic technique emerged, allowing the treatment of a large class of nontrivial problems. This technique is what nowadays is called the "Rieman-Hilbert" problem. In simple terms, it essentially amounts at finding an analytic function in the complex plane having a prescribed jump across a certain curve.

Several existence theorems for Riemann–Hilbert problems have been produced by Krein, Gohberg and others (see the book by Clancey and Gohberg [**23**], (1981)).

We recall that the first application of the Riemann–Hilbert method to integrable PDEs is found in the works of Manakov, Shabat, and Zakharov done in 1975-1979, and since then it has been widely used in soliton theory (see, e.g. [**68**]).

Another important class of applications of the Riemann–Hilbert approach deals with quantum exactly solvable models, starting with the work of the Japanese school in the 1980s (Jimbo, Miwa, Mori, and Sato), with other contributions and developments in the late 1980s and in the 1990s by Izergin, Korepin, Slavnov, Deift, Zhou, and Its (see the monograph [**54**] for details and references).

In 1991 Fokas, Its and Kitaev [**38**] established a fundamental connection between the Riemann–Hilbert approach and the theory of orthogonal polynomials and matrix models. This point of view was crucial in solving some of the long-standing problems in the asymptotics of orthogonal polynomials related to universalities in random matrices (see e.g. [**29**]). Given a weight on a contour, the corresponding orthogonal polynomials can be computed via the solution of a Riemann–Hilbert factorization problem. Furthermore, the distribution of eigenvalues of random matrices in several ensembles is reduced to computations involving orthogonal polynomials (see e.g. [**9**]).

The work of Fokas, Its and Kitaev was useful also to study random permutations. Along these lines, the most celebrated example of the RH approach applied to random permutations is the theorem of Baik, Deift and Johansson (1999) (see [**9**]) on the distribution of the length of the longest increasing sub-sequence of a random permutation of N numbers.

In general terms, a typical Riemann–Hilbert problem can be stated as follows.

Assume that Γ is an oriented contour in the complex z-plane. The orientation defined two sides, that we denote by $+$ and $-$. The contour Γ might have points of self-intersection, and a priori might have more than one connected component. Also, let $V(z)$ be a matrix function defined on the contour Γ, i.e. a map from Γ into the set of $N \times N$ invertible matrices. From the data (Γ, V), the Riemann–Hilbert problem consists of finding a matrix function $Y(z) \in \mathbb{C}^{N\times N}$ such that following conditions are satisfied.

(1) The function $Y(z)$ is analytic for $z \in \mathbb{C}/\Gamma$.
(2) The following jump condition is satisfied:

$$Y_+(z) = Y_-(z)V(z), \tag{11}$$

at all points of non-intersection in Γ, where $Y_+(z)$ and $Y_-(z)$ denote the non-tangential limits of $Y(z)$ as we approach Γ from the $+$ side and the $-$ side, respectively.

(3) As z tends to infinity along any direction outside Γ, $Y(z)$ tends to the identity matrix.

In the simplest case $V(z)$ is smooth and integrable. In more complicated cases it could have singularities. The limits $Y_+(z)$ and $Y_-(z)$ could be classical and continuous or they could be taken in the L_2 sense.

In our work, we have generalized the classical Riemann–Hilbert problem to the matrix case.

We have considered the cases when $\Gamma = \mathbb{R}$ and $\Gamma = \mathbb{T}$ (see Chapters 2 and 4, respectively). To be more concrete, let us illustrate the case of the matrix Riemann–Hilbert problem on the real line which will be of interest in our research. Precisely, we have solved the problem consisting in the determination of a $2N \times 2N$ matrix function $Y_n(z) \in \mathbb{C}^{2N\times 2N}$ such that

(1) $Y_n(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$.
(2) On $\mathbb{R}$, Y_n satisfies the jumping condition

$$Y_{n_+}(z) = Y_{n_-}(z) \begin{pmatrix} \mathbb{I}_N & \rho(x) \\ 0 & \mathbb{I}_N \end{pmatrix}.$$

(3) Asymptotically, it behaves as

$$Y_n(z) = (\mathbb{I}_{2N} + O(z^{-1})) \begin{pmatrix} \mathbb{I}_N z^n & 0 \\ 0 & \mathbb{I}_N z^{-n} \end{pmatrix} \qquad \text{for } z \to \infty.$$

We have the following result: The unique solution of the Riemann–Hilbert problem stated above is represented by the matrix function $Y_n(z)$

$$Y_n(z) := \begin{pmatrix} P_n(z) & Q_n(z) \\ c_{n-1}\tilde{P}_{n-1}(z) & c_{n-1}\tilde{Q}_{n-1}(z) \end{pmatrix}, \qquad n \geq 1, \tag{12}$$

expressed in terms of monic matrix orthogonal polynomials with respect to the measure $\rho(x)\,\mathrm{d}\,x$ and its Cauchy transforms.

From the matrix function $Y_n(z)$ one can construct other related Riemann–Hilbert problems for certain auxiliary matrix functions, involving again a matrix moment. The analysis of the recurrence relations satisfied by these matrix functions allows to define an Abelian algebra of matrix coefficients, which exists for any choice of the weight function $\rho(x) = \exp(-V(x))$.

One of the keys results (that will be proven in Chapter 2) is that the matrix orthogonal polynomials P_n, as well as their Cauchy transforms Q_n, satisfy the following recursion relations:

$$P_{n+1}(z) = zP_n(z) - \frac{1}{2}\beta_n P_{n-1}(z), \tag{13}$$

where the recursion coefficients β_n are given by

$$\beta_n := 2c_n^{-1}c_{n-1}, \qquad n \geq 1, \qquad \beta_0 := 0. \tag{14}$$

These relations hold in generality. In particular, if one restricts to the case of potentials

$$V(z) = \alpha z^2 + \mathbb{I}_n z^4, \qquad \alpha = \alpha^\dagger, \tag{15}$$

one obtains the following result: the recursion coefficients β_n given by eq. (14), for the class of Freud matrix orthogonal polynomials associated with the quartic potential (15), satisfy the relation

$$\beta_{n+1} = n\beta_n^{-1} - \beta_{n-1} - \beta_n - \alpha, \qquad n = 1, 2, \ldots$$

This matrix recurrence relation will be the main object of interest of Chapter 2, where its singularity confinement properties will be studied thoroughly.

We have obtained similar results for the case of the Riemann–Hilbert problem on the unit circle $\mathbb{T}$.

In particular, we are led to a recurrence of the form

$$\alpha_{n+1} = -(n+1)k^{-1}(\mathbb{I}_N - \alpha_n^2)^{-1}\alpha_n - \alpha_{n-1},$$

with suitable initial conditions. This equation can be regarded as a matrix form of the discrete version of the second Painlevé equation.

4. Singularity confinement

The Painlevé transcendents were introduced more than a century ago as a result of a longstanding research aimed at identifying nonlinear ordinary differential equations whose solutions are free of movable critical points (the so called Painlevé property) (see [**70**] and [**24**] for a recent account of the state of the art in this subject).

The Painlevé equations are of special relevance in Mathematical Physics, and in 2D Quantum Gravity and Topological Field Theory[**24, 38, 31**].

A discrete version of the Painlevé property, named the *singularity confinement property*, was introduced by Grammaticos, Ramani and Papageorgiou in 1991 [**46**].

These authors focused on several discrete models, and in particular on a discrete version of the first Painlevé equation (dPI). For the dPI they discovered that if eventually a singularity appears at some specific value of the discrete independent variable, then it disappears after performing few steps or iterations in the equation.

This approach, known as *singularity confinement*, can be considered as the analog of the Painlevé property [**70**] for discrete equations. Ramani, Hietarinta and Grammaticos also derived some discrete versions of the other five Painlevé equations [**72, 73**], such as a discrete version of Painlevé II (dPII). See also the interesting papers [**48**] and [**59**]. Another important contribution came from Van Assche [**8**], who also derived discrete versions of the Painlevé equations.

The discovery of the *Painlevé property* for ODEs allowed to relate the notion of *integrability* with the local analysis of movable isolated singularities of solutions of dynamical systems [**24**]. We also mention that the Painlevé property was extended to the case of PDEs (method of Weiss-Tabor-Carnevale).

4.1. Painlevé property. Given a linear ODE, its general solution has no singularity which depends on the constants of integration.

The case of nonlinear ODEs is more challenging. The possibility of defining a new function from a solution of a nonlinear ODE only relies on the nature of the singularities of the solution.

There exist two classifications of singularities of the solutions of ODEs: on one hand to be critical or non critical (i.e. to display local multivaluedness or local singlevaluedness around a singularity), on the other hand to be fixed (their location does not depend on the initial conditions) or movable (the opposite).

For single-valuedness, the only worry is that a singularity be at the same time movable and critical, since one then does not know where to put a cut in order to remove the multivaluedness.

We shall say that an ODE possesses the Painlevé property (PP) if its general solution has no movable critical singularities. [**25**]

Note that essential singularities are not involved in the definition of the PP. The group of invariance of the PP is the class of transformations

$$u(x) = (\alpha(x)U(X) + \beta(x))/(\gamma(x)U(X) + \delta(x)), \quad X = \xi(x),$$

$\alpha\delta - \beta\gamma \neq 0$, in which α, β, γ, δ, ξ are arbitrary analytic functions. It depends on four arbitrary functions. In order to find new functions, one must (i) investigate nonlinear ODEs of order one, then two, then three ...;

(ii) select those which possess the PP, (iii) prove whether their general solution defines a new (or an old) function. This ambitious program yields various intermediate outputs, which we now outline.

5. Statement of the main Results

The main results of this work can be summarized as follows.

1) We have generalized and solved the Riemann–Hilbert problem to the case of matrix orthogonal polynomials. We considered the case of matrix Freud polynomials, both on the real line and on the unit circle.

2) Novel matrix integrable models have been derived. They can be considered to be generalizations of the discrete Painlevé equations I and II, which in their scalar versions have been proposed by Van Assche.

3) A thorough analysis of the singularity confinement properties of the matrix discrete Painlevé equation I obtained by solving the Riemann–Hilbert problem on the line has been performed. We have shown analytically that the property holds generically, i.e. for a large set of initial conditions. The set of conditions where it does not hold is represented by specific algebraic varieties in the space of parameters.

In short, our analysis paves the way to a formulation of a theory of matrix integrable discrete models, still largely unexplored. Interestingly enough, the standard approaches used in soliton theory, in particular the Riemann–Hilbert approach and the discrete version of the Painlevé singularity analysis, keep playing a crucial role in the matrix theory we wish to develop. Our research represent a first exploration of this new exciting field.

6. Future Perspectives

In this work, a close relation has been established among matrix Riemann-Hilbert problems, matrix discrete models and the singularity confinement property. Our results suggest that matrix models can be naturally obtained by extending conveniently, to the matrix case, the usual mathematical techniques developed in the last decades in the study of integrable systems.

Many problems and aspects originating from the present work deserve further investigation. Here we mention several possible research lines.

i) The prominent role of orthogonal polynomials in our analysis naturally leads to the idea of exploring classes of polynomials different with respect to those considered in the present Thesis. For instance, in the article in preparation *Non-Hermitian matrix models and Matrix Freud Orthogonal Polynomials on the Circle*, by G. Cassatella Contra, M. Mañas and P. Tempesta, a non-Hermitian version of the theory of Freud matrix orthogonal polynomials on the circle is under analysis. This family of polynomials is related to a non-Hermitian matrix of measures, in the spirit of recent work in the theory of random matrices.

ii) We are also planning to study a non-abelian generalization of our matrix Rieman–Hilbert approach. A priori, it could be obtained, for instance, by working with non-abelian versions of the matrix potentials characterizing orthogonal polynomials of Freud type.

iii) A Lax formulation of the matrix models proposed in this Thesis would confirm the close relation between the singularity confinement property and the notion of integrability for matrix systems.

Bibliography

[1] M. J. Ablowitz and J. F. Ladik, *Nonlinear differential-difference equations* J. Math. Phys. 16, 598 (1975).

[2] R. Abraham, J. E. Marsden, *Foundations of Mechanics*, 2nd ed., The Benjamin/Cummings Publishing Company, Reading (1978).

bibitemaclarkson M. Ablowitz and Ladik, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, (1991).

[3] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions With Formulas, Graphs and Mathematical Tables*, National Bureau of Standards, Washington (1964).

[4] H. Akaike, *Block Toeplitz matrix inversion*, SIAM J. Appl. Math., 24 (1977) 234-241.

[5] D. V. Anosov and A. A. Bolibruch, *The Riemann–Hilbert Problem*, Aspects of Mathematics, E22, Braunschweig, (1994).

[6] A. I. Aptekarev and E. M. Nikishin, *The scattering problem for a discrete Sturm–Liouville operator*, Math. USSR-Sb. 49, 325-355 (1984).

[7] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Encyclopaedia of Mathematical Sciences, Springer (2006).

[8] W. Van Assche, *Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials*, Proceedings of the International Conference on Difference Equations, special Functions and Orthogonal Polynomials, World Scientific (2007) 687–725.

[9] J. Baik, P. Deift and K. Johansson, *On the distribution of the length of the longest increasing subsequence of random permutations*, J. Amer. Math. Soc. 12 (1999) 1119–1178.

[10] Yu. M. Berezanskii, *Expansions in eigenfunctions of self-adjoint operators*, Transl. Math. Monographs 17, Amer. Math. Soc. (1968).

[11] A. I. Bobenko and Y. Suris, *Discrete differential geometry. Integrable structure*, Graduate Studies in Mathematics, 98, American Mathematical Society, Providence, RI (2008).

[12] A. I. Bobenko and Y. Suris, *Integrable systems on quad–graphs*, Int. Math. Res. Not. 11 (2002) 573–611.

[13] J. Borrego, M. Castro, A. J. Durán, *Orthogonal matrix polynomials satisfying differential equations with recurrence coefficients having non-scalar limits*, `arXiv:1102.1578v1`.

[14] J. Boussinesq, *Essai sur la theorie des eaux courantes, Memoires presentes par divers savants*, Academie des Sciences d'Institut National de France, XXIII, (1877) 1–680

[15] J. P. Burg, *Maximum entropy spectral analysis*, Ph.D. dissertation, Geophysics Dep., Stanford Umversity, Stanford, CA, (1975).

[16] M.J. Cantero, L. Moral and L. Velázquez, *Differential properties of matrix orthogonal polynomials*, J. Concrete Appl. Math. 3, 313–334.

[17] G. Carlet, B. Dubrovin and Y. Zhang, *The extended Toda hierarchy*, Mosc. Math. J. 4 (2004) 313–332.

[18] G. A. Cassatella-Contra and M. Mañas, *Riemann–Hilbert problems, matrix orthogonal polynomals and discrete matrix equations with singularity confinement*, Stud. Appl. Math. 128 (2012) 252–274.

[19] G. Cassatella Contra, M. Mañas and P. Tempesta, *Singularity confinement for matrix discrete Painlevé equations*, Nonlinearity 27, 2321-2335, 2014

[20] G. Cassatella Contra, M. Mañas and P. Tempesta, *Freud polynomials on the circle and a matrix Painlevé II discrete equation*, Preprint 2015.

[21] P. Chebyshev, *Théorie de mécnismes connus sous le nom de parallélogrammes*. Mem. Acad. sci. St.-Pétersb. 7 (1854) 539-564.

[22] P. Chebyshev, *Sur les questions de minima se rattachent à la représention approximative des fonctions*. Bull. Acad. sci. St.-Pétersb. Cl. phys.-math. 16 (1858) 145-149; Mem. Acad. sci. St.-Pétersb. Ser. 7, 1 (1859) 199-291.

[23] K. F. Clancey, I. Gohberg, *Factorization of Matrix Functions and Singular Integral Operators*, Operator Theory Adv. Appl., vol. 3, Birkhäuser, Basel, (1981).

[24] R. Conte, *The Painlevé Property. One Century Later*, Springer–Verlag, New York (1999).

[25] R. Conte and M. Musette, *The Painlevé handbook* Springer, Berlin, (2008). http://www.springer.com/physics/book/978-1-4020-8490-4 Russian translation: http://shop.rcd.ru/details/1304

[26] R. D. Costin, *Matrix valued polynomials generated by the scalar-type Rodrigues' formulas*, Journal of Approximation Theory 161 (2009) 693–705.

[27] D. Damanik, A. Pushnitski, and B. Simon, *The Analytic Theory of Matrix Orthogonal Polynomials*, Surveys in Approximation Theory 4, (2008) 1–85 and also `arXiv:0711.2703`.

[28] P. A. Deift, *Orthogonal Polynomials and Random Matrices. A Riemann–Hilbert Approach*, Courant Lecture Notes in Mathematics, vol. 3, CIMS, New York (1999).

[29] P. A. Deift, A. R. Its and X. Zhou, *Long-time asymptotics for integrable nonlinear wave equations, Important Developments in Soliton Theory (A. S. Fokas and V. E. Zakharov, eds.)*, Springer-Verlag (1993) 181–204.

[30] Ph. Delsarte, Y. Genin, Y. Kamp, *Orthogonal polynomial matrices on the unit circle*, IEEE Trans. Circuits Systems, 25 (3) (1978) 149–160.

[31] B. Dubrovin, Geometry of 2D topological field theories. In: *Integrable Systems and Quantum Groups* (Authors: R. Donagi, B. Dubrovin, E. Frenkel, E. Previato), Eds. M. Francaviglia, S.Greco, Springer Lecture Notes in Math., **1620** (1996), pp. 120-348.

[32] A. J. Durán, *Matrix inner product having a matrix symmetric second order differential operator*, Rocky Mountain Journal of Mathematics, 27 (1997) 585–600.

[33] A. J. Durán and M. D. de la Iglesia, *Second order differential operators having several families of orthogonal matrix polynomials as eigenfunctions*, Int. Math. Research Notices, (2008) Article ID rnn084, 24 pages.

[34] A. J. Durán and F. J. Grünbaum, *Orthogonal matrix polynomials satisfying second order differential equations*, Internat. Math. Res. Notices 10 (2004) 461–484.

[35] A. J. Durán and F. J. Grünbaum, *Orthogonal matrix polynomials, scalar-type Rodrigues' formulas and Pearson equations*, Journal of Approximation Theory 134 (2005) 267–280.

[36] A. J. Durán and F. J. Grünbaum, *Structural formulas for orthogonal matrix polynomials satisfying second order differential equations*, I, Constr. Approx. 22 (2005) 255–271.

[37] J. Favard, *Sur les meilleurs procédés d'approximation de certaines classes de fonctions par des polynomes trigonométriques*, Bull. sci. math., 61 (1937), 209-224, 243-256.

[38] A. S. Fokas, A. R. Its and A. V. Kitaev, *Discrete Painlevé equations and their appearance in quantum gravity*, Comm. Math. Phys. 142, no. 2, (1991) 313–344

[39] A. S. Fokas, A. R. Its and A. V. Kitaev, *The isomonodromy approach to matrix models in 2D quantum gravity*, Commun. Math. Phys. 147 (1992) 395–430.

[40] G. Freud, *On the coefficients in the recursion formulae of orthogonal polynomials*, Proc. Royal Irish Acad. A76, (1976) 1–6.

[41] R. Gambini and J. Pullin, *Canonical Quantization of General Relativity in Discrete Space–Times*, Phys. Rev. Lett. 90 (2003) 021301.

[42] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Method for Solving the Korteweg-deVries Equation*, Phys. Rev.Lett. 19 (1967) 1095–1097.

[43] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Korteweg-de Vries equation and generalization. VI. Methods for exact solutions*, Comm. Pure Appl. Math. 27 (1974) 97–133.

[44] J. S. Geronimo, *Scattering theory and matrix orthogonal polynomials on the real line*, Circuits Systems Signal Process 1 (1982) 471–495.

[45] J.S. Geronimo, *Matrix orthogonal polynomials on the unit circle*, J. Math. Phys. 22 (1981) 1359–1365.

[46] B. Grammaticos, A. Ramani and V. Papageorgiou, *Do integrable mappings have the Painlevé property?*, Phys. Rev. Lett 67 (1991) 1825–1828.

[47] H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables*, Acta Math. 99 (1958) 165–202.

[48] J. Hietarinta and C. Viallet, *Discrete Painlevé I and singularity confinement in projective space*, Chaos Solitons and Fractals 11 (2000) 29–32.

[49] G. 't Hooft, *Quantization of point particles in (2+1)–dimensional gravity and spacetime discreteness*, Class. Quantum Grav. 13 (1996) 1023–1039.

[50] A. R. Its, The Riemann-Hilbert Problem and Integrable Systems, Notices of the AMS, 1389-1400 (2003).

[51] T. Kailath, *A view of three decades of linear filtering theory*, IEEE Trans. Inform Theory, 20 (1974) 146–181.

[52] T. Kailath, A. Vieira, and M. Morf, *Inverses of Toeplitz operators, innovations and orthogonal polynomials*, IEEE Con. J. Decision Contr. (1975) 749-754.

[53] B. Konopelchenko, *Discrete integrable systems and deformations of associative algebras*, J. Phys. A 42 (2009) 454003.

[54] V. E. Korepin, N. M. Bogoliubov, A. G. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge Monographs on Mathematical Physics (1993).

[55] D. J. Korteweg and G. de Vries , *On the Change of Form of Long Waves Advancing in a Rectangular Canal and on a New Type of Long Stationary Waves*, Phil. Mag. 39 (1895) 422–443.

[56] M. G. Krein, *Infinite J-matrices and a matrix moment problem*, Dokl. Akad. Nauk. SSSR 69, 2 (1949) 125–128.

[57] M. G. Krein, *Fundamental aspects of the representation theory of hermitian operators with deficiency index (m,m)*, AMS Translations, Series 2, 97, Providence, Rhode Island (1971) 75–143.

[58] M. G. Krein, W. M. Smirnov, and A. N. Kolmogorov, *On a generalization of some investigations of G. Szego*, Dokl. Akad. Nauk SSSR 46 (1945), 91–94.

[59] S. Lafortune and A. Goriely, *Singularity confinement and algebraic integrability*, J. Math. Phys. 45 (2004) 1191–1208.

[60] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Commun. Pure Appl. Math 21 (1968) 467-490.

[61] D. Levi, P. Tempesta and P. Winternitz, *Lorentz and Galilei invariance on lattices*, Phys. Rev. D 69 (2004) 105011.

[62] D. Levi and P. Winternitz, *Continuous symmetries of difference equations*, J. Phys. A 9 (2006) R1–R63.

[63] A. P. Magnus, *Freud's equations for orthogonal polynomials as discrete Painlevé equations*, Symmetries and Integrability of Difference Equations (Canterbury, 1996), London Math. Soc. Lecture Note Ser., 255, Cambridge University Press (1999) 228–243.

[64] F. Marcellán, E. Godoy, *Orthogonal polynomials on the unit circle: distribution of zeros*, Journal of Computational and Applied Mathematics, Volume 37, Issues 1–3, 18 (1991) 195–208.

[65] V. A. Marchenko, *Sturm-Liouville Operators and Applications*, Birkhauser, Basel (1986).

[66] A. S. Mischenko and A. T. Fomenko, *Generalized Liouville method of integration of Hamiltonian systems*, Funct. Anal. Appl. **12**, 133 (1978).

[67] M. Morf, A. Vieira, and T. Kailath, *Covariance characterization by partial autocorrelation matrices*, Annals of Statistics (1978).

[68] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of Solitons*, Consultants Bureau, New York (1984).

[69] S. P. Novikov, A. S. Shvarts, *Discrete Lagrangian systems on graphs. Symplecto–topological properties* Uspekhi Mat. Nauk 54, no.1 325 (1999) 257–258. Translation in Russian Math. Surveys 54, no. (1999) 258–259.

[70] P. Painlevé, *Leçons sur la théorie analytique des equations différentielles (Leçons de Stockholm, delivered in 1895)*, Hermann, Paris (1897). Reprinted in *Œvres de Paul Painlevé, vol. I, Éditions du CNRS*, Paris (1973).

[71] E. Parzen, *Multiple time-series modeling*, Proc. Second Znt. Symp. Multiu. Analysis, Dayton, OH (1968).

[72] A. Ramani, B. Grammaticos and J. Hietarinta, *Discrete versions of the Painlevé equations*, Phys. Rev. Lett. 67 (1991) 1829–1832.

[73] A. Ramani, D. Takahashi, B. Grammaticos and Y. Ohta, *The ultimate discretisation of the Painlevé equations*, Physica D, 114, Issues 3–4, (1998) 185–196.

[74] C. Rovelli and L. Smolin, *Discreteness of area and volume in quantum gravity*, Nucl. Phys. B 442, (1995) 593–619.

[75] E. K. Sklyanin, *Separation of variables: new trends*, Prog. Theor. Phys. Suppl. **118**, 35–60 (1995).

[76] T. J. Stieljes, *Quelques recherches sur la théorie des quadratures dites mécaniques*, Ann. sci. ëcole norm. spér., Sér,. 3.1 (1884) 409-426.

[77] T. J. Stieljes, *Recherches sur les fractions continues*, Ann. fac. sci. Univ Toulouse, 8 (1894) 1-122.

[78] G. Szegő, *Beiträge zur Theorie der Toeplitzschen Formen*, Mathematische Zeitschrift Springer Berlin, Heidelberg 6 (1920) 167–202.

[79] G. Szegő, *Beiträge zur Theorie der Toeplitzschen Formen*, Mathematische Zeitschrift Springer Berlin, Heidelberg 9 (1921) 167–190,

[80] G. Szegő, *Orthogonal Polynomials*, Colloquium Publications, XXIII, American Mathematical Society, Providence (1975).

[81] Yu. B. Suris, *The Problem of Integrable Discretization: Hamiltonian Approach*, Progress in Mathematics, 219 Basel, Birkhäuser (2003).

[82] P. Tempesta, P. Winternitz, J. Harnad, W. Miller, Jr, G. Pogosyan, M. A. Rodríguez (eds), *Superintegrability in Classical and Quantum Systems*, Montréal, CRM Proceedings and Lecture Notes, AMS, vol. **37** (2004).

[83] M. Toda, *Theory of Nonlinear Lattices*, Springer Series in Solid State Sciences 20, Springer-Verlag, Berlin (1989).

[84] N. J. Zabusky and M. D. Kruskal, *Interaction of Solitons in a Collisionless Plasma and the Recurrence of Initial States*, Phys. Rev. Lett. 16 (1965) 240–243.

[85] V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one dimensional of waves in nonlinearmedia* , Sov. Phys. JETP 34 (1972) 62-69.

[86] V. E. Zakharov and A. B. Shabat, *A scheme for integrating the nonlinear equations of mathematical physics by the method of inverse scattering method*, Func. Anal. Appl. 8 (1974) 226-235.

CHAPTER II

RIEMANN-HILBERT PROBLEMS, MATRIX ORTHOGONAL POLYNOMIALS AND DISCRETE MATRIX EQUATIONS WITH SINGULARITY CONFINEMENT

Riemann–Hilbert Problems, Matrix Orthogonal Polynomials and Discrete Matrix Equations with Singularity Confinement

By G.A. Cassatella-Contra and M. Mañas[*]

In this paper, matrix orthogonal polynomials in the real line are described in terms of a Riemann–Hilbert problem. This approach provides an easy derivation of discrete equations for the corresponding matrix recursion coefficients. The discrete equation is explicitly derived in the matrix Freud case, associated with matrix quartic potentials. It is shown that, when the initial condition and the measure are simultaneously triangularizable, this matrix discrete equation possesses the singularity confinement property, independently if the solution under consideration is given by the recursion coefficients to quartic Freud matrix orthogonal polynomials or not.

1. Introduction

The study of singularities of the solutions of nonlinear ordinary differential equations and, in particular, the quest of equations whose solutions are free of movable critical points, the so called Painlevé property, lead, more than 100 years ago, to the Painlevé transcendents, see [1] (and [2] for a recent account of the state of the art in this subject). The Painlevé equations are relevant in a diversity of fields, not only in Mathematics but also, for example, in Theoretical Physics and in particular in 2D Quantum Gravity and Topological Field Theory, see for example, [2].

[*]Address for correspondence: M. Mañas, Departamento de Física Teórica II, Universidad Compltense, 28040-Madrid, Spain; e-mail: manuel.manas@fis.ucm.es

DOI: 10.1111/j.1467-9590.2011.00541.x

STUDIES IN APPLIED MATHEMATICS 128:252–274

A discrete version of the Painlevé property, the singularity confinement property, was introduced for the first time by Grammaticos et al. in 1991 [3], when they studied some discrete equations, including the dPI equation (discrete version of the first Painlevé equation), see also the contribution of these authors to [2]. For this equation they realized that if eventually a singularity could appear at some specific value of the discrete independent variable it would disappear after performing few steps or iterations in the equation. This property, as mentioned previously, is considered by these authors as the equivalent of the Painlevé property [1] for discrete equations. Ramani et al. also derived some discrete versions of the other five Painlevé equations [4, 5]. See also the interesting papers [6] and [7].

Freud orthogonal polynomials in the real line [8] are associated to the weight

$$w_\rho(x) = |x|^\rho e^{-|x|^m}, \quad \rho > -1, \quad m > 0.$$

Interestingly, for $m = 2, 4, 6$ it has been shown [9] that from the recursion relation

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x),$$

the orthogonality of the polynomials leads to a recursion relation satisfied by the recursion coefficients a_n. In particular, for $m = 4$ Van Assche obtains for a_n the discrete Painlevé I equations, and therefore its singularities are confined. For related results see also [10]. For a modern and comprehensive account of this subject see the survey [11].

In 1992, it was found [12] that the solution of a 2×2 Riemann–Hilbert problem can be expressed in terms of orthogonal polynomials in the real line and its Cauchy transforms. Later on this property has been used in the study of certain properties of asymptotic analysis of orthogonal polynomials and extended to other contexts, for example, for the multiple orthogonal polynomials of mixed type [13].

Orthogonal polynomials with matrix coefficients on the real line have been considered in detail first by Krein [14, 15] in 1949, and then were studied sporadically until the last decade of the twentieth century. These are some papers of this subject: Berezanskii (1968) [16], Geronimo (1982) [17], and Aptekarev and Nikishin (1984) [18]. In the last paper they solved the scattering problem for a kind of discrete Sturm–Liouville operators that are equivalent to the recurrence equation for scalar orthogonal polynomials. They found that polynomials that satisfy a recurrence relation of the form

$$x P_k(x) = A_k P_{k+1}(x) + B_k P_k(x) + A^*_{k-1} P_{k-1}(x), \qquad k = 0, 1, \dots,$$

are orthogonal with respect to a positive definite measure. This is a matricial version of Favard's theorem for scalar orthogonal polynomials. Then, in the 1990s and the 2000s some authors found that matrix orthogonal polynomials (MOP) satisfy in certain cases some properties that satisfy scalar valued orthogonal polynomials; for example, Laguerre, Hermite and Jacobi polynomials, that is, the scalar-type Rodrigues' formula [19–21] and a second-order differential equation [22–24].

Later on, it has been proven [25] that operators of the form $D=\partial^2 F_2(t)+\partial^1 F_1(t)+\partial^0 F_0$ have as eigenfunctions different infinite families of MOP's. Moreover, in [24] a new family of MOP's satisfying second-order differential equations whose coefficients do not behave asymptotically as the identity matrix was found. See also [26].

The aim of this paper is to explore the singularity confinement property in the realm of MOP. For that aim following [12] we formulate the matrix Riemann–Hilbert problem associated with the MOP's. From the Riemann–Hilbert problem it follows not only the recursion relations but also, for a type of matrix Freud weight with $m=4$, a nonlinear recursion relation, Equation (58), for the matrix recursion coefficients, that might be considered a matrix version—non-Abelian—of the discrete Painlevé I. Finally, we prove that this matrix equation possesses the singularity confinement property, and that after a maximum of 4 steps the singularity disappears. This happens when the quartic potential V and the initial recursion coefficient are simultaneously triangularizable. It is important to notice that the recursion coefficients for the matrix orthogonal Freud polynomials provide solutions to Equation (58) and therefore the singularities are confined. A relevant fact for this solution is that the collection of all recursion coefficients is an Abelian set of matrices. However, not all solutions of Equation (58) define a commutative set; nevertheless, the singularity confinement still holds. In this respect we must stress that our singularity confinement proof do not rely in MOP theory but only on the analysis of the discrete equation. This special feature is not present in the scalar case previously studied elsewhere.

The layout of this paper is as follows. In Section 2 the Riemann–Hilbert problem for MOP is derived and some of its consequences studied. In Section 3 a discrete matrix equation, for which the recursion coefficients of the Freud MOP's are solutions, is derived and it is also proven that its singularities are confined. Therefore, it might be considered as a matrix discrete Painlevé I equation.

2. Riemann–Hilbert problems and matrix orthogonal polynomials in the real line

2.1. Preliminaries on monic matrix orthogonal polynomials in the real line

A family of MOP's in the real line [11] is associated with a matrix-valued measure μ on $\mathbb{R}$; that is, an assignment of a positive semi-definite $N\times N$ Hermitian matrix $\mu(X)$ to every Borel set $X\subset\mathbb{R}$ which is countably additive.

However, in this paper we constraint ourselves to the following case: given an $N \times N$ Hermitian matrix $V(x) = (V_{i,j}(x))$, we choose $\mathrm{d}\mu = \rho(x)\mathrm{d}x$, being $\mathrm{d}x$ the Lebesgue measure in $\mathbb{R}$, and with the weight function specified by $\rho = \exp(-V(x))$ (thus ρ is a positive semi-definite Hermitian matrix). Moreover, we will consider only even functions in x, $V(x) = V(-x)$; in this situation the finiteness of the measure $\mathrm{d}\mu$ is achieved for any set of polynomials $V_{i,j}(x)$ in x^2. Associated with this measure we have a unique family $\{P_n(x)\}_{n=0}^{\infty}$ of monic MOP

$$P_n(z) = \mathbb{I}_N z^n + \gamma_n^{(1)} z^{n-1} + \cdots + \gamma_n^{(n)} \in \mathbb{C}^{N\times N},$$

such that

$$\int_{\mathbb{R}} P_n(x) x^j \rho(x)\mathrm{d}x = 0, \qquad j = 0, \ldots, n-1. \tag{1}$$

Here $\mathbb{I}_N$ denotes the identity matrix in $\mathbb{C}^{N\times N}$.

In terms of the moments of the measure $\mathrm{d}\mu$,

$$m_j := \int_{\mathbb{R}} x^j \rho(x)\mathrm{d}x \in \mathbb{C}^{N\times N}, \qquad j = 0, 1, \ldots,$$

we define the truncated moment matrix

$$m^{(n)} := (m_{i,j}) \in \mathbb{C}^{nN\times nN},$$

with $m_{i,j} = m_{i+j}$ and $0 \leq i, j \leq n-1$. Invertibility of $m^{(n)}$, that is, $\det m^{(n)} \neq 0$, is equivalent to the existence of a unique family of monic MOP. In fact, we can write Equation 1 as

$$\begin{pmatrix} m_0 & \cdots & m_{n-1} \\ \vdots & & \vdots \\ m_{n-1} & \cdots & m_{2n-2} \end{pmatrix} \begin{pmatrix} \gamma_n^{(n)} \\ \vdots \\ \gamma_n^{(1)} \end{pmatrix} = \begin{pmatrix} -m_n \\ \vdots \\ -m_{2n-1} \end{pmatrix}, \tag{2}$$

and hence uniqueness is equivalent to $\det m^{(n)} \neq 0$. From the uniqueness and evenness we deduce that

$$P_n(z) = \mathbb{I}_N z^n + \gamma_n^{(2)} z^{n-2} + \gamma_n^{(4)} z^{n-4} + \cdots + \gamma_n^{(n)}, \tag{3}$$

where $\gamma_n^{(n)} = 0$ if n is odd.

The Cauchy transform of $P_n(z)$ is defined by

$$Q_n(z) := \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{P_n(x)}{x - z} \rho(x)\mathrm{d}x, \tag{4}$$

which is analytic for $z \in \mathbb{C}\backslash\mathbb{R}$. Recalling $\frac{1}{z-x} = \frac{1}{z}\sum_{j=0}^{n-1}\frac{x^j}{z^j} + \frac{1}{z}\frac{(\frac{x}{z})^n}{1-\frac{x}{z}}$ and Equation 1 we get

$$Q_n(z) = -\frac{1}{2\pi\mathrm{i}}\frac{1}{z^{n+1}}\int_{\mathbb{R}}\frac{P_n(x)\,x^n}{1-\dfrac{x}{z}}\rho(x)\,\mathrm{d}x, \tag{5}$$

and consequently

$$Q_n(z) = c_n^{-1}z^{-n-1} + O(z^{-n-2}), \qquad z \to \infty, \tag{6}$$

where we have introduced the coefficients

$$c_n := \left(-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} P_n(x)\rho(x)x^n\mathrm{d}x\right)^{-1}, \tag{7}$$

relevant in the sequel of the paper.

PROPOSITION 1. *We have that c_n satisfies*

$$\det c_n = -2\pi\mathrm{i}\frac{\det(m^{(n)})}{\det(m^{(n+1)})}. \tag{8}$$

Proof. To prove it just define $\boldsymbol{m} := (m_n, \dots, m_{2n-1})$, consider the identity

$$\begin{pmatrix} m^{(n)^{-1}} & 0 \\ 0 & \mathbb{I}_N \end{pmatrix} m^{(n+1)} = \begin{pmatrix} \mathbb{I}_{nN} & m^{(n)^{-1}}\boldsymbol{m} \\ \boldsymbol{m}^t & m_{2n} \end{pmatrix},$$

and apply the Gauss elimination method to get

$$\frac{\det(m^{(n+1)})}{\det(m^{(n)})} = \det(m_{2n} - \boldsymbol{m}^t m^{(n)^{-1}}\boldsymbol{m}) \neq 0;$$

from Equation (2) we conclude $m_{2n} - \boldsymbol{m}^t m^{(n)^{-1}}\boldsymbol{m} = \int_{\mathbb{R}} P_n(x)x^n\rho(x)\mathrm{d}x$. □

The evenness of V leads to $Q_n(z) = (-1)^{n+1}Q_n(-z)$, so that

$$Q_n(z) = c_n^{-1}z^{-n-1} + \sum_{j=2}^{\infty} a_n^{(2j-1)}z^{-n-2j+1}, \qquad z \to \infty. \tag{9}$$

In particular,

$$Q_0(z) = c_0^{-1}z^{-1} + c_1^{-1}z^{-3} + O(z^{-5}), \quad z \to \infty. \tag{10}$$

Finally, if we assume that $V_{i,j}$ are Hölder continuous we get the Plemelj formulae

$$(Q_n(z)_+ - Q_n(z)_-)|_{x\in\mathbb{R}} = P_n(x)\rho(x), \tag{11}$$

with $Q_n(x)_+ = Q_n(z)|_{z=x+i0^+}$ and $Q_n(x)_- = Q_n(z)|_{z=x+i0^-}$. ■

2.2. Riemann–Hilbert problem

Definition 1. *The Riemann–Hilbert problem to consider here is the finding of a $2N \times 2N$ matrix function $Y_n(z) \in \mathbb{C}^{2N\times 2N}$ such that*

1. *$Y_n(z)$ is analytic in $z \in \mathbb{C}\backslash\mathbb{R}$.*
2. *Asymptotically behaves as*

$$Y_n(z) = (\mathbb{I}_{2N} + O(z^{-1}))\begin{pmatrix} \mathbb{I}_N z^n & 0 \\ 0 & \mathbb{I}_N z^{-n} \end{pmatrix}, \qquad z \to \infty. \tag{12}$$

3. *On $\mathbb{R}$ we have the jump*

$$Y_n(x)_+ = Y_n(x)_- \begin{pmatrix} \mathbb{I}_N & \rho(x) \\ 0 & \mathbb{I}_N \end{pmatrix}. \tag{13}$$

An easy extension of the connection among orthogonal polynomials in the real line with a particular Riemann–Hilbert problem discovered in [12] can be proven in this matrix context.

Proposition 2. *The unique solution to the Riemann–Hilbert problem specified in Definition 1 is given in terms of monic matrix orthogonal polynomials with respect to $\rho(x)\mathrm{d}x$ and its Cauchy transforms:*

$$Y_n(z) = \begin{pmatrix} P_n(z) & Q_n(z) \\ c_{n-1}P_{n-1}(z) & c_{n-1}Q_{n-1}(z) \end{pmatrix}, \quad n \geq 1. \tag{14}$$

Proof. In the first place let us show that $\left(\begin{smallmatrix} P_n(z) & Q_n(z) \\ c_{n-1}P_{n-1}(z) & c_{n-1}Q_{n-1}(z) \end{smallmatrix}\right)$ does satisfy the three conditions requested by Definition 1.

1. As the matrix orthogonal polynomials P_n are analytic in $\mathbb{C}$ and its Cauchy transforms are analytic in $\mathbb{C}\backslash\mathbb{R}$, the proposed solution is analytic in $\mathbb{C}\backslash\mathbb{R}$.
2. Replacing the asymptotics of the matrix orthogonal polynomials and its Cauchy transforms we get $\left(\begin{smallmatrix} P_n(z) & Q_n(z) \\ c_{n-1}P_{n-1}(z) & c_{n-1}Q_{n-1}(z) \end{smallmatrix}\right) \to \left(\begin{smallmatrix} z^n + O(z^{n-1}) & O(z^{-n-1}) \\ O(z^{n-1}) & z^{-n} + O(z^{-n-1}) \end{smallmatrix}\right) = (\mathbb{I}_{2N} + O(z^{-1}))\left(\begin{smallmatrix} \mathbb{I}_N z^n & 0 \\ 0 & \mathbb{I}_N z^{-n} \end{smallmatrix}\right)$ when $z \to \infty$.
3. From Equation (11) we get $\left(\begin{smallmatrix} P_n(x+\mathrm{i}0) & Q_n(x+\mathrm{i}0) \\ c_{n-1}P_{n-1}(x+\mathrm{i}0) & c_{n-1}Q_{n-1}(x+\mathrm{i}0) \end{smallmatrix}\right) - \left(\begin{smallmatrix} P_n(x-\mathrm{i}0) & Q_n(x-\mathrm{i}0) \\ c_{n-1}P_{n-1}(x-\mathrm{i}0) & c_{n-1}Q_{n-1}(x-\mathrm{i}0) \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & P_n(x)\rho(x) \\ 0 & c_{n-1}P_{n-1}(x)\rho(x) \end{smallmatrix}\right)$.

Then, a solution to the RH problem is $Y_n = \left(\begin{smallmatrix} P_n(z) & Q_n(z) \\ c_{n-1}P_{n-1}(z) & c_{n-1}Q_{n-1}(z) \end{smallmatrix}\right)$. But the solution is unique, as we will show. Given any solution Y_n, its determinant

$\det Y_n(z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$ and satisfies

$$\begin{aligned}\det Y_n(x)_+ &= \det\left(Y_n(x)_-\begin{pmatrix}\mathbb{I}_N & \rho(x)\\ 0 & \mathbb{I}_N\end{pmatrix}\right) = \det Y_n(x)_- \det\begin{pmatrix}\mathbb{I}_N & \rho(x)\\ 0 & \mathbb{I}_N\end{pmatrix}\\ &= \det Y_n(x)_-.\end{aligned}$$

Hence, $\det Y_n(z)$ is analytic in $\mathbb{C}$. Moreover, Definition 1 implies that

$$\det Y_n(z) = 1 + O(z^{-1}), \qquad z \to \infty,$$

and Liouville theorem ensures that

$$\det Y_n(z) = 1, \quad \forall z \in \mathbb{C}. \tag{15}$$

From Equation (15) we conclude that Y_n^{-1} is analytic in $\mathbb{C}\backslash\mathbb{R}$. Given two solutions Y_n and $\tilde{Y}_n$ of the RH problem we consider the matrix $\tilde{Y}_n Y_n^{-1}$, and observe that from property 3 of Definition 1 we have $(\tilde{Y}_n Y_n^{-1})_+ = (\tilde{Y}_n Y_n^{-1})_-$, and consequently $\tilde{Y}_n Y_n^{-1}$ is analytic in $\mathbb{C}$. From Definition 1 we get $\tilde{Y}_n Y_n^{-1} \to \mathbb{I}_{2N}$ as $z \to \infty$, and Liouville theorem implies that $\tilde{Y}_n Y_n^{-1} = \mathbb{I}_{2N}$; that is, $\tilde{Y}_n = Y_n$ and the solution is unique. ■

DEFINITION 2. *Given the matrix Y_n we define*

$$S_n(z) := Y_n(z)\begin{pmatrix}\mathbb{I}_N z^{-n} & 0\\ 0 & \mathbb{I}_N z^n\end{pmatrix}. \tag{16}$$

PROPOSITION 3.

1. *The matrix S_n has unit determinant:*

$$\det S_n(z) = 1. \tag{17}$$

2. *It has the special form*

$$S_n(z) = \begin{pmatrix}A_n(z^2) & z^{-1}B_n(z^2)\\ z^{-1}C_n(z^2) & D_n(z^2)\end{pmatrix}. \tag{18}$$

3. *The coefficients of S_n admit the asymptotic expansions*

$$\begin{aligned}A_n(z^2) &= \mathbb{I}_N + S^{(2)}_{n,11}z^{-2} + O(z^{-4}), & B_n(z^2) &= S^{(1)}_{n,12} + S^{(3)}_{n,12}z^{-2} + O(z^{-4}),\\ C_n(z^2) &= S^{(1)}_{n,21} + S^{(3)}_{n,21}z^{-2} + O(z^{-4}), & D_n(z^2) &= \mathbb{I}_N + S^{(2)}_{n,22}z^{-2} + O(z^{-4}),\end{aligned}$$
for $z \to \infty$. (19)

Proof.

1. Is a consequence of Equations (15) and (16).
2. It follows from the parity of P_n and Q_n.

3. Equation (12) implies the following asymptotic behaviour

$$S_n(z) = \mathbb{I}_{2N} + S_n^{(1)} z^{-1} + O(z^{-2}), \qquad z \to \infty, \tag{20}$$

and Equation (18) gives

$$S_n^{(2i)} = \begin{pmatrix} S_{n,11}^{(2i)} & 0 \\ 0 & S_{n,22}^{(2i)} \end{pmatrix}, \quad S_n^{(2i-1)} = \begin{pmatrix} 0 & S_{n,12}^{(2i-1)} \\ S_{n,21}^{(2i-1)} & 0 \end{pmatrix},$$

and the result follows. □

Observe that from Equation (18) we get

$$S_n^{-1}(z) = \begin{pmatrix} \tilde{A}_n(z^2) & z^{-1}\tilde{B}_n(z^2) \\ z^{-1}\tilde{C}_n(z^2) & \tilde{D}_n(z^2) \end{pmatrix}, \tag{21}$$

with the asymptotic expansions for $z \to \infty$

$$\begin{aligned} \tilde{A}_n(z^2) &= \mathbb{I}_N + \big(S_{n,12}^{(1)} S_{n,21}^{(1)} - S_{n,11}^{(2)}\big) z^{-2} + O(z^{-4}), \\ \tilde{B}_n(z^2) &= -S_{n,12}^{(1)} - \big(S_{n,12}^{(3)} - S_{n,11}^{(2)} S_{n,12}^{(1)} + S_{n,12}^{(1)}\big(S_{n,21}^{(1)} S_{n,12}^{(1)} - S_{n,22}^{(2)}\big)\big) z^{-2} \\ &\quad + O(z^{-4}), \\ \tilde{C}_n(z^2) &= -S_{n,21}^{(1)} + \big(-S_{n,21}^{(3)} + S_{n,21}^{(1)} S_{n,11}^{(2)} + \big(S_{n,22}^{(2)} - S_{n,21}^{(1)} S_{n,12}^{(1)}\big) S_{n,21}^{(1)}\big) z^{-2} \\ &\quad + O(z^{-4}), \\ \tilde{D}_n(z^2) &= \mathbb{I}_N + \left(S_{n,21}^{(1)} S_{n,12}^{(1)} - S_{n,22}^{(2)}\right) z^{-2} + O(z^{-4}). \end{aligned}$$

■

2.2.1. Recursion relations. We now introduce the necessary elements, within the Riemann–Hilbert problem approach, to derive the recursion relations and properties of the recursion coefficients in the context of matrix orthogonal polynomials.

DEFINITION 3. *We introduce the matrix*

$$Z_n(z) := Y_n(z) \begin{pmatrix} \rho(z) & 0 \\ 0 & \mathbb{I}_N \end{pmatrix} = \begin{pmatrix} P_n(z)\rho(z) & Q_n(z) \\ c_{n-1} P_{n-1}(z)\rho(z) & c_{n-1} Q_{n-1}(z) \end{pmatrix}. \tag{22}$$

PROPOSITION 4.

1. *$Z_n(z)$ is analytic on $\mathbb{C} \backslash \mathbb{R}$,*
2. *for $z \to \infty$ it holds that*

$$Z_n(z) = (\mathbb{I}_{2N} + O(z^{-1})) \begin{pmatrix} z^n \rho(z) & 0 \\ 0 & z^{-n} \mathbb{I}_N \end{pmatrix}, \tag{23}$$

3. *over $\mathbb{R}$ it is satisfied*

$$Z_n(x)_+ = Z_n(x)_- \begin{pmatrix} \mathbb{I}_N & \mathbb{I}_N \\ 0 & \mathbb{I}_N \end{pmatrix}. \tag{24}$$

DEFINITION 4. *We introduce*

$$M_n(z) := \frac{\mathrm{d}Z_n(z)}{\mathrm{d}z} Z_n^{-1}(z), \tag{25}$$

$$R_n(z) := Z_{n+1}(z) Z_n{}^{-1}(z) = Y_{n+1}(z) Y_n{}^{-1}(z). \tag{26}$$

We can easily show that

PROPOSITION 5. *The matrices M_n and R_n satisfy*

$$M_{n+1}(z) R_n(z) = \frac{\mathrm{d}}{\mathrm{d}z} R_n(z) + R_n(z) M_n(z). \tag{27}$$

Proof. It follows from the compatibility condition

$$T \frac{\mathrm{d}Z_n(z)}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z} T Z_n(z),$$

where $T F_n := F_{n+1}$.

We can also show that

PROPOSITION 6. *For the functions $R_n(z)$ and $M_n(z)$ we have the expressions*

$$R_n(z) = \begin{pmatrix} z\mathbb{I}_N & -S^{(1)}_{n,12} \\ S^{(1)}_{n+1,21} & 0 \end{pmatrix}, \tag{28}$$

$$M_n(z) = \left[\begin{pmatrix} A_n(z^2) \dfrac{\mathrm{d}\rho(z)}{\mathrm{d}z} \rho^{-1}(z) \tilde{A}_n(z^2) & A_n(z^2) z^{-1} \dfrac{\mathrm{d}\rho(z)}{\mathrm{d}z} \rho^{-1}(z) \tilde{B}_n(z^2) \\ z^{-1} C_n(z^2) \dfrac{\mathrm{d}\rho(z)}{\mathrm{d}z} \rho^{-1}(z) \tilde{A}_n(z^2) & z^{-2} C_n(z^2) \dfrac{\mathrm{d}\rho(z)}{\mathrm{d}z} \rho^{-1}(z) \tilde{B}_n(z^2) \end{pmatrix} \right]_+, \tag{29}$$

where $[\cdot]_+$ denotes the part in positive powers of z.

Proof. The expression for R_n is a consequence of the following reasoning:

1. In the first place notice that $R_n(z)$ is analytic for $z \in \mathbb{C} \backslash \mathbb{R}$.
2. Moreover, denoting

$$R_{n+}(x) := Y_{n+1_+}(x) (Y_{n+}(x))^{-1}, \tag{30}$$

$$R_{n-}(x) := Y_{n+1-}(x)(Y_{n-}(x))^{-1}, \tag{31}$$

and substituting Equation (13) in Equation (30) we get $R_{n+}(x) = R_{n-}(x)$ and therefore $R_n(z)$ is analytic in $\mathbb{C}$.

(3) Finally, if we substitute Equation (16) in Equation (26) we deduce that

$$\begin{aligned} R_n(z) &= Y_{n+1}(z)Y_n{}^{-1}(z) \\ &= S_{n+1}(z)\begin{pmatrix} z\mathbb{I}_N & 0 \\ 0 & z^{-1}\mathbb{I}_N \end{pmatrix} S_n^{-1}(z) \\ &= \begin{pmatrix} z\mathbb{I}_N & 0 \\ 0 & 0 \end{pmatrix} + S_{n+1}^{(1)}\begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & z^{-1}\mathbb{I}_N \end{pmatrix} - \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & 0 \end{pmatrix} S_n^{(1)} + O(z^{-1}), \\ & z \to \infty, \end{aligned}$$

and the analyticity of R_n in $\mathbb{C}$ leads to the desired result.

For the expression for M_n we have the argumentation:

1. $M_n(z)$ is analytic for $z \in \mathbb{C}\backslash\mathbb{R}$.
2. Given

$$M_{n+}(x) := \frac{\mathrm{d}Z_{n+}(x)}{\mathrm{d}z}(Z_{n+}(x))^{-1}, \tag{32}$$

$$M_{n-}(x) := \frac{\mathrm{d}Z_{n-}(x)}{\mathrm{d}z}(Z_{n-}(x))^{-1}. \tag{33}$$

Substituting Equation (24) in Equation (32) we get

$$M_{n+}(x) = M_{n-}(x),$$

and therefore $M_n(z)$ is analytic over $\mathbb{C}$.

3. From Equations (16) and (22) we see that $Z_n(z)$ is

$$Z_n(z) = S_n(z)\begin{pmatrix} z^n\rho(z) & 0 \\ 0 & z^{-n}\mathbb{I}_N \end{pmatrix}, \tag{34}$$

so that

$$\frac{\mathrm{d}Z_n(z)}{\mathrm{d}z}Z_n^{-1}(z) = \frac{\mathrm{d}S_n(z)}{\mathrm{d}z}S_n(z)^{-1} + S_n(z)K_n(z)S_n^{-1}(z), \tag{35}$$

where

$$K_n(z) := \begin{pmatrix} nz^{-1}\mathbb{I}_N + \dfrac{\mathrm{d}\rho(z)}{\mathrm{d}z}\rho^{-1}(z) & 0 \\ 0 & -nz^{-1}\mathbb{I}_N \end{pmatrix}.$$

Finally, as $M_n(z)$ is analytic over $\mathbb{C}$, Equation (35) leads to

$$M_n(z) = \frac{\mathrm{d}Z_n(z)}{\mathrm{d}z} Z_n^{-1}(z) = \left[S_n(z) \begin{pmatrix} \dfrac{\mathrm{d}\rho(z)}{\mathrm{d}z}\rho^{-1}(z) & 0 \\ 0 & 0 \end{pmatrix} S_n^{-1}(z) \right]_+ . \tag{36}$$

Observe that the diagonal terms of M_n are odd functions of z while the off diagonal are even functions of z. Now we give a parametrization of the first coefficients of S in terms of c_n. ■

PROPOSITION 7. *The following formulae hold true*

$$\begin{aligned}
&S^{(1)}_{n,12} = c_n^{-1}, \quad S^{(1)}_{n,21} = c_{n-1},\\
&S^{(2)}_{n,11} = -\sum_{i=1}^{n} c_i^{-1} c_{i-1} + c_n^{-1} c_{n-1}, \quad S^{(2)}_{n,22} = \sum_{i=1}^{n} c_{i-1} c_i^{-1},\\
&S^{(3)}_{n,21} = -c_{n-1} \sum_{i=1}^{n-1} c_i^{-1} c_{i-1} + c_{n-2}, \quad S^{(3)}_{n,12} = c_n^{-1} \sum_{i=1}^{n+1} c_{i-1} c_i^{-1}.
\end{aligned}$$

Proof. Equating the expressions for $Y_n(z)$ provided by Equation (14) and Equation (16) we get

$$\begin{aligned}
Y_n(z) &= \begin{pmatrix} P_n(z) & Q_n(z) \\ c_{n-1}P_{n-1}(z) & c_{n-1}Q_{n-1}(z) \end{pmatrix}\\
&= \begin{pmatrix} z^n \mathbb{I}_N & 0 \\ 0 & z^{-n}\mathbb{I}_N \end{pmatrix} \big(\mathbb{I}_{2N} + S^{(1)}_n z^{-1} + S^{(2)}_n z^{-2} + S^{(3)}_n z^{-3} + O(z^{-4})\big),\\
&\quad z \to \infty.
\end{aligned}$$

Expanding the right hand side we get

$$\begin{aligned}
&S^{(1)}_{n,21} = c_{n-1}, \quad S^{(1)}_{n,12} = c_n^{-1},\\
&S^{(2)}_{1,11} = 0,\\
&S^{(3)}_{1,21} = S^{(3)}_{2,21} = 0, \quad S^{(3)}_{n,21} = c_{n-1} S^{(2)}_{n-1,11}, \quad n \geq 2,\\
&S^{(3)}_{n,12} = c_n^{-1} S^{(2)}_{n+1,22},
\end{aligned} \tag{37}$$

where we have used that

$$S^{(2)}_{1,22} = c_0 c_1^{-1}, \tag{38}$$

which can be proved from Equation (10). Introducing Equation (37) into Equation (28) we get

$$R_n(z) = \begin{pmatrix} z\mathbb{I}_N & -c_n^{-1} \\ c_n & 0 \end{pmatrix}, \tag{39}$$

and Equations (16) and (39) lead to

$$S_{n+1}(z) = \begin{pmatrix} z\mathbb{I}_N & -c_n^{-1} \\ c_n & 0 \end{pmatrix} S_n(z) \begin{pmatrix} z^{-1}\mathbb{I}_N & 0 \\ 0 & z\mathbb{I}_N \end{pmatrix},$$

so that

$$S^{(2)}_{n+1,11} - S^{(2)}_{n,11} = -c_n^{-1} c_{n-1}, \tag{40}$$

$$S^{(3)}_{n,12} - c_n^{-1} S^{(2)}_{n,22} = c_{n+1}^{-1}, \tag{41}$$

where we have used Equation (37). From Equations (37) and (41) we get

$$S^{(2)}_{n+1,22} - S^{(2)}_{n,22} = c_n c_{n+1}^{-1}. \tag{42}$$

Summing up in n in Equations (40) and (42) we deduce

$$\sum_{i=1}^{n-1} \left(S^{(2)}_{i+1,11} - S^{(2)}_{i,11}\right) = -\sum_{i=1}^{n-1} c_i^{-1} c_{i-1},$$

$$\sum_{i=1}^{n-1} \left(S^{(2)}_{i+1,22} - S^{(2)}_{i,22}\right) = \sum_{i=1}^{n-1} c_i c_{i+1}^{-1},$$

which leads to

$$S^{(2)}_{n,11} = -\sum_{i=1}^{n} c_i^{-1} c_{i-1} + c_n^{-1} c_{n-1}, \tag{43}$$

$$S^{(2)}_{n,22} = \sum_{i=1}^{n} c_{i-1} c_i^{-1}, \tag{44}$$

where we have used Equations (37) and (38). Finally Equations (37), (43) and (44) give

$$S^{(3)}_{n,21} = -c_{n-1} \sum_{i=1}^{n-1} c_i^{-1} c_{i-1} + c_{n-2}, \tag{45}$$

valid for $n \geq 2$, and

$$S^{(3)}_{n,12} = c_n^{-1} \sum_{i=1}^{n+1} c_{i-1} c_i^{-1}. \tag{46}$$

Notice that Equation (37) gives

$$S^{(3)}_{1,21} = 0. \tag{47}$$

■

PROPOSITION 8. *Matrix orthogonal polynomials P_n (and its Cauchy transforms Q_n) are subject to the following recursion relations*

$$P_{n+1}(z) = zP_n(z) - \frac{1}{2}\beta_n P_{n-1}(z), \tag{48}$$

with the recursion coefficients β_n given by

$$\beta_n := 2c_n^{-1}c_{n-1}, \quad n \geq 1, \quad \beta_0 := 0. \tag{49}$$

Proof. Observe that Equation (26) can be written as

$$Y_{n+1}(z) = R_n(z)Y_n(z). \tag{50}$$

Then, if we replace Equations (14) and (39) into Equation (50) we get the result.

We now show some commutative properties of the polynomials and the recursion coefficients. ■

PROPOSITION 9. *Let $f(z) : \mathbb{C} \to \mathbb{C}^{N\times N}$ such that $[V(x), f(z)] = 0$ $\forall (x, z) \in \mathbb{R} \times \mathbb{C}$, then*

$$[c_n, f(z)] = [\beta_n, f(z)] = 0, \quad n \geq 0, \quad \forall z \in \mathbb{C},$$
$$[P_n(z'), f(z)] = 0, \quad n \geq 0, \quad \forall z, z' \in \mathbb{C}.$$

Proof. Let us suppose that for a given $m \geq 0$ we have

$$[P_m(x), f(z)] = [P_{m-1}(x), f(z)] = 0. \tag{51}$$

Then, recalling Equation (7) these expressions give

$$[c_m, f(z)] = [c_{m-1}, f(z)] = 0, \tag{52}$$

respectively. Therefore, using the recursion relations Equations (48) and (49) we obtain

$$[P_{m+1}(x), f(z)] = x[P_m(x), f(z)] - \left[c_m^{-1}c_{m-1}P_{m-1}(x), f(z)\right] = 0.$$

This means that

$$[c_{m+1}, f(z)] = 0. \tag{53}$$

Hypothesis Equation (51) holds for $m = 1$, consequently $[c_n, f(z)] = 0$ for $n = 0, 1, \ldots$ and Equation (49) implies $[\beta_n, f(z)] = 0$. Finally, as the coefficients of the matrix orthogonal $P_n(z)$ are polynomials in the β's we conclude that $[P_n(z'), f(z)] = 0$ for all $z, z' \in \mathbb{C}$. ■

COROLLARY 1. *Suppose that* $[V(x), V(z)] = 0$ *for all* $x \in \mathbb{R}$ *and* $z \in \mathbb{C}$, *then*

$$[P_n(z), P_m(z')] = 0, \quad \forall n, m \geq 0, \quad z, z' \in C, \tag{54}$$

$$[c_n, c_m] = 0, \tag{55}$$

$$[\beta_n, \beta_m] = 0. \tag{56}$$

Proof. Applying Proposition 9 to $f = V$ we deduce that $[P_n(z'), V(z)] = 0$, so that it allows to use again Proposition 9 but now with $f = P_n$ and get the stated result. From Equations (7) and (54) we deduce Equations (55) and using Equation (49) we get Equation (56). ■

3. A discrete matrix equation, related to Freud matrix orthogonal polynomials, with singularity confinement

We will consider the particular case when

$$V(z) = \alpha z^2 + \mathbb{I}_N z^4, \quad \alpha = \alpha^\dagger. \tag{57}$$

Observe that $[V(z), V(z')] = 0$ for any pair of complex numbers z, z'. Hence, in this case the corresponding set of MOP $\{P_n\}_{n=0}^{\infty}$, that we refer as matrix Freud polynomials, is an Abelian set. Moreover, we have

$$[c_n, c_m] = [\beta_n, \beta_m] = [c_n, \alpha] = [\beta_n, \alpha] = 0, \quad \forall n, \quad m = 0, 1, \ldots.$$

In this situation, we have

THEOREM 1. *The recursion coefficients* β_n *Equation* (49) *for the Freud matrix orthogonal polynomials determined by Equation* (57) *satisfy*

$$\beta_{n+1} = n\beta_n^{-1} - \beta_{n-1} - \beta_n - \alpha, \quad n = 1, 2, \ldots, \tag{58}$$

with $\beta_0 := 0$.

Proof. We compute now the matrix M_n, for which we have

$$M_n(z) = \left[\begin{pmatrix} -A_n(z^2)\big(2\alpha z + 4z^3\mathbb{I}_N\big)\tilde{A}_n(z^2) & -A_n(z^2)\big(2\alpha + 4z^2\mathbb{I}_N\big)\tilde{B}_n(z^2) \\ -C_n(z^2)\big(2\alpha + 4z^2\mathbb{I}_N\big)\tilde{A}_n(z^2) & -C_n(z^2)\big(2\alpha z^{-1} + 4z\mathbb{I}_N\big)\tilde{B}_n(z^2) \end{pmatrix}\right]_+, \tag{59}$$

and is clear that

$$M_n(z) = M_n^{(3)}z^3 + M_n^{(2)}z^2 + M_n^{(1)}z + M_n^{(0)}, \tag{60}$$

with

$$M_n^{(3)}=\begin{pmatrix}-4\mathbb{I}_N & 0\\ 0 & 0\end{pmatrix},\ M_n^{(2)} = \begin{pmatrix}0 & 4S_{n,12}^{(1)}\\ -4S_{n,21}^{(1)} & 0\end{pmatrix},$$

$$M_n^{(1)}=\begin{pmatrix}-2\alpha - 4S_{n,12}^{(1)}S_{n,21}^{(1)} & 0\\ 0 & 4S_{n,21}^{(1)}S_{n,12}^{(1)}\end{pmatrix},$$

$$M_n^{(0)}=\begin{pmatrix}0 & \begin{matrix}2\alpha S_{n,12}^{(1)} + 4S_{n,12}^{(3)}\\ \quad + 4S_{n,12}^{(1)}\big(S_{n,21}^{(1)}S_{n,12}^{(1)} - S_{n,22}^{(2)}\big)\end{matrix}\\ \begin{matrix}-2S_{n,21}^{(1)}\alpha - 4S_{n,21}^{(3)}\\ \quad + 4S_{n,21}^{(1)}S_{n,11}^{(2)} - 4S_{n,21}^{(1)}S_{n,12}^{(1)}S_{n,21}^{(1)}\end{matrix} & 0\end{pmatrix}.$$

Replacing Equations (37)–(46) into Equation (60) we get

$$\begin{aligned}
M_n^{(3)}&=\begin{pmatrix}-4\mathbb{I}_N & 0\\ 0 & 0\end{pmatrix}, \quad M_n^{(2)} = \begin{pmatrix}0 & 4c_n^{-1}\\ -4c_{n-1} & 0\end{pmatrix},\\
M_n^{(1)}&=\begin{pmatrix}-2\alpha - 4c_n^{-1}c_{n-1} & 0\\ 0 & 4c_{n-1}c_n^{-1}\end{pmatrix},
\end{aligned} \tag{61}$$

$$M_1^{(0)} = \begin{pmatrix}0 & 4c_2^{-1} + 4c_1^{-1}c_0c_1^{-1} + 2\alpha c_1^{-1}\\ -4c_0c_1^{-1}c_0 - 2c_0\alpha & 0\end{pmatrix}, \tag{62}$$

$$M_n^{(0)}= \begin{pmatrix}0 & 4c_{n+1}^{-1} + 4c_n^{-1}c_{n-1}c_n^{-1} + 2\alpha c_n^{-1}\\ -4c_{n-2} - 4c_{n-1}c_n^{-1}c_{n-1} - 2c_{n-1}\alpha & 0\end{pmatrix},$$

$$n \geq 2. \tag{63}$$

The compatibility condition Equation (27) together with Equations (39) and (60)–(63) gives

$$\begin{aligned}
&4\big(c_{n+2}^{-1}c_n + c_{n+1}^{-1}c_nc_{n+1}^{-1}c_n - c_n^{-1}c_{n-1}c_n^{-1}c_{n-1} - c_n^{-1}c_{n-2}\big) + 2\alpha c_{n+1}^{-1}c_n\\
&\quad - 2c_n^{-1}c_{n-1}\alpha = \mathbb{I}_N,
\end{aligned}$$

for $n \geq 2$ and

$$4\left(c_3^{-1}c_1 + c_2^{-1}c_1c_2^{-1}c_1 - c_1^{-1}c_0c_1^{-1}c_0\right) + 2\alpha c_2^{-1}c_1 - 2c_1^{-1}c_0\alpha = \mathbb{I}_N,$$

which can be written as

$$\beta_{n+2}\beta_{n+1} + \beta_{n+1}^2 - \beta_n^2 - \beta_n\beta_{n-1} + \alpha\beta_{n+1} - \beta_n\alpha = \mathbb{I}_N \tag{64}$$

for $n \geq 2$ and

$$\beta_3\beta_2 + \beta_2^2 - \beta_1^2 + \alpha\beta_2 - \beta_1\alpha = \mathbb{I}_N, \tag{65}$$

respectively. Using the Abelian character of the set of β's we arrive to

$$\beta_{n+2}\beta_{n+1} + \beta_{n+1}^2 - \beta_n^2 - \beta_n\beta_{n-1} + \alpha(\beta_{n+1} - \beta_n) = \mathbb{I}_N, \quad n=2, 3, \ldots, \tag{66}$$

$$\beta_3\beta_2 + \beta_2^2 - \beta_1^2 + \alpha(\beta_2 - \beta_1) = \mathbb{I}_N. \tag{67}$$

Summing up in Equation (66) from $i = 2$ up to $i = n$ we obtain

$$\sum_{i=2}^{n}\left[\beta_{i+2}\beta_{i+1} + \beta_{i+1}^2 - \beta_i^2 - \beta_i\beta_{i-1} + \alpha(\beta_{i+1} - \beta_i)\right] = \sum_{i=2}^{n}\mathbb{I}_N, \tag{68}$$

and consequently we conclude that

$$\beta_{n+2}\beta_{n+1} + \beta_{n+1}\beta_n + {\beta_{n+1}}^2 + \alpha\beta_{n+1} = n\mathbb{I}_N + k, \quad n \geq 1, \tag{69}$$

where

$$k := \beta_2\beta_1 + \beta_3\beta_2 + \beta_2^2 + \alpha\beta_2 - \mathbb{I}_N = \beta_2\beta_1 + \beta_1^2 + \beta_1\alpha, \tag{70}$$

where we have used Equation (67). We now proceed to show that $k = \mathbb{I}_N$. Equation (25) implies, for $n = 1$ and $z = 0$,

$$Z_1'(0) = M_1^{(0)}Z_1(0), \tag{71}$$

with $M_1^{(0)}$ given in Equation (62). This leads to

$$\begin{pmatrix} P_1'(0) \\ c_0P_0'(0) \end{pmatrix} = M_1^{(0)}\begin{pmatrix} P_1(0) \\ c_0P_0(0) \end{pmatrix}. \tag{72}$$

Now, using Equation (3) we deduce that

$$\begin{pmatrix} \mathbb{I}_N \\ 0 \end{pmatrix} = M_1^{(0)}\begin{pmatrix} 0 \\ c_0 \end{pmatrix}, \tag{73}$$

which allows us to immediately deduce that

$$\beta_2\beta_1 + \beta_1^2 + \beta_1\alpha = \mathbb{I}_N, \tag{74}$$

and consequently $k = \mathbb{I}_N$. Finally, we get

$$\beta_{n+2}\beta_{n+1} + \beta_{n+1}\beta_n + {\beta_{n+1}}^2 + \alpha\beta_{n+1} = n\mathbb{I}_N + \mathbb{I}_N. \tag{75}$$

Finally, notice that Equation (74) reads

$$\beta_2 = \beta_1^{-1} - \beta_1 - \alpha. \tag{76}$$

This theorem ensures that β_1 fixes β_n for all $n \geq 2$, and therefore $\beta_n = \beta_n(\beta_1, \alpha)$. Moreover, we will see now that the solutions β_n not only commute with each other but also that they can be simultaneously conjugated to lower matrices. This result is relevant in our analysis of the confinement of singularities. ■

3.1. On singularity confinement

The study of the singularities of the discrete matrix equations Equation (58) reveals, as we will show, that they are confined; that is, the singularities may appear eventually, however they disappear in few steps, no more than four. The mentioned singularities in Equation (58) appear when for some n the matrix β_n is not invertible, that is $\det \beta_n = 0$, and we can not continue with the sequence. However, for a better understanding of this situation in the discrete case we just request that $\det \beta_n$ is a small quantity so that β_n^{-1} and β_{n+1} exist, but they are very "large" matrices in some appropriate sense. To be more precise we will consider a small parameter ϵ and suppose that in a given step m of the sequence we have

$$\beta_{m-1} = O(1), \qquad \det \beta_{m-1} = O(1), \tag{77}$$

$$\beta_m = O(1), \qquad \det \beta_m = O(\epsilon^r), \tag{78}$$

for $\epsilon \to 0$ and with $r \leq N - 1$. In other words, we have the asymptotic expansions

$$\beta_{m-1} = \beta_{m-1,0} + \beta_{m-1,1}\epsilon + O(\epsilon^2), \quad \epsilon \to 0, \quad \det \beta_{m-1,0} \neq 0, \tag{79}$$

$$\beta_m = \beta_{m,0} + \beta_{m,1}\epsilon + O(\epsilon^2), \qquad \epsilon \to 0, \qquad \dim \operatorname{Ran} \beta_{\mathrm{m},0} = N - r. \tag{80}$$

We now proceed with some preliminar material. In particular we show that we can restrict the study to the triangular case.

Proposition 10. *Let us suppose that β_1 and α are simultaneously triangularizable matrices; that is, there exist an invertible matrix M such that $\beta_1 = M\phi_1 M^{-1}$ and $\alpha = M\gamma M^{-1}$ with ϕ_1 and γ lower triangular matrices. Then, the solutions β_n of Equation* (58) *can be written as*

$$\beta_n = M\phi_n M^{-1}, \quad n \geq 0,$$

where ϕ_n, $n = 0, 1, \ldots,$ are lower triangular matrices satisfying

$$\phi_{n+1} = n\phi_n^{-1} - \phi_{n-1} - \phi_n - \gamma.$$

Moreover, let us suppose that for some integer m the matrices β_{m+1}, β_m and α are simultaneously triangularizable, then all the sequence $\{\beta_n\}_{n=0}^{\infty}$ is simultaneously triangularizable.

Proof. On one hand, from Equation (58) we conclude that $M^{-1}\beta_2 M$ is lower triangular and in fact that $\{M^{-1}\beta_n M\}_{n\geq 0}$ is a sequence of lower triangular matrices. On the other hand, if for some integer m the matrices β_{m+1}, β_m, and α are simultaneously triangularizable we have

$$\begin{aligned}\beta_{m+1} &= m\beta_m^{-1} - \beta_m - \beta_{m-1} - \alpha,\\ \beta_m &= (m-1)\beta_{m-1}^{-1} - \beta_{m-1} - \beta_{m-2} - \alpha,\end{aligned}$$

which implies that β_{m-1}, β_{m-2} are triangularized by the same transformation that triangularizes β_{m+1}, β_m, and α.

The simultaneous triangularizability can be achieved, for example, when $[\beta_1, \alpha] = 0$, as in this case we can always find an invertible matrix M such that $\beta_1 = M\phi_1 M^{-1}$ and $\alpha = M\gamma M^{-1}$ where ϕ_1 and γ are lower triangular matrices, for example, by finding the Jordan form of these two commuting matrices. This is precisely the situation for the solutions related with matrix orthogonal polynomials. Obviously, this is just a sufficient condition. From now on, and following Proposition 10 , we will assume that the simultaneous triangularizability of α and β_1 holds and study the case in where α and all the β's are lower triangular matrices. Thus, we will use the splitting

$$\beta_n = D_n + N_n, \tag{81}$$

$$\alpha = \alpha_D + \alpha_N, \tag{82}$$

where $D_n = \operatorname{diag}(D_{n;1}, \ldots, D_{n;N})$ and $\alpha_D = \operatorname{diag}(\alpha_{D,1}, \ldots, \alpha_{D,N})$ are the diagonal parts of β_n and α, respectively, and N_n and α_N are the strictly lower parts of β_n and α, respectively. Then, Equation (58) splits into

$$\begin{aligned}D_{n+1} + N_{n+1} = nD_n^{-1} - D_{n-1} - D_n - \alpha_D\\ + n\bar{N}_n - N_{n-1} - N_n - \alpha_N,\end{aligned} \tag{83}$$

where $\bar{N}_n$ denotes the strictly lower triangular part of β_n^{-1}; that is,

$$\beta_n^{-1} = D_n^{-1} + \bar{N}_n.$$

Hence, Equation (58) decouples into

$$D_{n+1} = nD_n^{-1} - D_{n-1} - D_n - \alpha_D, \tag{84}$$

$$N_{n+1} = n\bar{N}_n - N_{n-1} - N_n - \alpha_N. \tag{85}$$

In this context, it is easy to realize that there always exists a transformation leading to the situation in where

$$\beta_{m,0}=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ \beta_{m,0;r+1,1} & \beta_{m,0;r+1,2} & \cdots & \beta_{m,0;r+1,r+1} & 0 & \cdots & 0\\ \beta_{m,0;r+2,1} & \beta_{m,0;r+2,2} & \cdots & \beta_{m,0;r+2,r+1} & \beta_{m,0;r+2,r+2} & \cdots & 0\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ \beta_{m,0;N,1} & \beta_{m,0;N,2} & \cdots & \beta_{m,0;N,r+1} & \beta_{m,0;N,r+2} & \cdots & \beta_{m,0;N,N} \end{pmatrix}. \tag{86}$$

■

PROPOSITION 11. *The singularities of the diagonal part are confined. More explicitly, if we assume that Equations* (79), (80), *and* (86) *hold true at a given step m then*

$$\begin{aligned} D_{m+1;i} &= \frac{m}{\beta_{m,1;i,i}}\epsilon^{-1}-\beta_{m-1,0;i,i}-\frac{\beta_{m,2;i,i}m}{\beta^2_{m,1;i,i}}-\alpha_{D,i}+O(\epsilon),\\ D_{m+2;i} &= -\frac{m}{\beta_{m,1;i,i}}\epsilon^{-1}+\beta_{m-1,0;i,i}+\frac{\beta_{m,2;i,i}m}{\beta^2_{m,1;i,i}}+O(\epsilon),\\ D_{m+3;i} &= -\beta_{m,1;i,i}\frac{m+3}{m}\epsilon+O(\epsilon^2), \end{aligned} \tag{87}$$

$$D_{m+4;i}=\frac{m\beta_{m-1,0;i,i}-2\alpha_{D,i}}{m+3}+O(\epsilon), \tag{88}$$

when $\epsilon\to 0$.

Proof. From Equations (79), (80), and (86) we deduce that

$$\begin{aligned} D_{m-1,i} &= \beta_{m-1,0;i,i}+\beta_{m-1,1;i,i}\epsilon+O(\epsilon^2),\\ D_{m,i} &= \beta_{m,1;i,i}\epsilon+O(\epsilon^2), \end{aligned}$$

for $\epsilon\to 0$, with $i\le r$. Substitution of these expressions in Equation (84) leads to the stated formulae. For $i\ge r+1$ the coefficients $D_{m-1;i}$ and $D_{m;i}$ are $O(1)$ as $\epsilon\to 0$, thus they do not vanish, and consequently there is confinement of singularities for the diagonal part D_n.

In what follows we will consider asymptotic expansions taking values in the set of lower triangular matrices

$$\mathbb{T} := \{T_0 + T_1\epsilon + O(\epsilon^2),\ \epsilon \to 0, \quad T_i \in \mathfrak{t}_N\}, \quad \mathfrak{t}_N := \{T = (T_{i,j}) \in \mathbb{C}^{N\times N},$$
$$X_{i,j} = 0 \text{ when } i > j\}, \tag{89}$$

where $\mathfrak{t}_N$ is the set of lower triangular $N \times N$ matrices. The reader should notice that this set $\mathbb{T} = \mathfrak{t}_N[[\epsilon]]$ is a subring of the ring of $\mathbb{C}^{N\times N}$-valued asymptotic expansions; in fact is a subring with identity, the matrix $\mathbb{I}_N$. We will use the notation

$$T_i := \begin{pmatrix} T_{i,11} & 0 \\ T_{i,21} & T_{i,22} \end{pmatrix}, \quad i \geq 1, \tag{90}$$

where $T_{i,11} \in \mathfrak{t}_r$, $T_{i,22} \in \mathfrak{t}_{N-r}$, and $T_{i,21} \in \mathbb{C}^{(N-r)\times r}$. We consider two sets of matrices determined by Equation (86), namely

$$\mathfrak{k} := \left\{K_0 = \begin{pmatrix} 0 & 0 \\ K_{0,21} & K_{0,22} \end{pmatrix}, K_{0,21} \in \mathbb{C}^{(N-r)\times r}, K_{0,22} \in \mathfrak{t}_{N-r}\right\},$$

$$\mathfrak{l} \ := \{L_{-1} = \begin{pmatrix} L_{-1,11} & 0 \\ L_{-1,21} & 0 \end{pmatrix},\ L_{-1,11} \in \mathfrak{t}_r, L_{-1,21} \in \mathbb{C}^{(N-r)\times r}\Big\},$$

and the related sets

$$\mathbb{K} := \left\{K = K_0 + K_1\epsilon + O(\epsilon^2) \in \mathbb{T}, \quad K_0 \in \mathfrak{k}\right\}, \tag{91}$$

$$\mathbb{L} := \left\{L = L_{-1}\epsilon^{-1} + L_0 + L_1\epsilon + O(\epsilon^2) \in \epsilon^{-1}\mathbb{T}, \quad L_{-1} \in \mathfrak{l}\right\}, \tag{92}$$

which fulfill the following important properties. ■

PROPOSITION 12.

1. *Both $\mathbb{K}$ and $\epsilon\mathbb{L}$ are subrings of the ring with identity $\mathbb{T}$, however these two subrings have no identity.*
2. *If an element $X \in \mathbb{K}$ with $r = r_0$ is such that* $\det X = O(\epsilon^{r_0})$ *for $\epsilon \to 0$, then $X^{-1} \in \mathbb{L}$, and reciprocally if $X \in \mathbb{L}$ with $r = r_0$ is such that* $\det X = O(\epsilon^{-r_0})$ *for $\epsilon \to 0$, then $X^{-1} \in \mathbb{K}$.*
3. *The subrings $\epsilon\mathbb{L}$ and $\mathbb{K}$ are bilateral ideals of* $\mathbb{T}$; *that is, $\mathbb{L} \cdot \mathbb{T} \subset \mathbb{L}$, $\mathbb{T} \cdot \mathbb{L} \subset \mathbb{L}$, $\mathbb{T} \cdot \mathbb{K} \subset \mathbb{K}$, and $\mathbb{K} \cdot \mathbb{T} \subset \mathbb{K}$.*
4. *We have $\mathbb{L} \cdot \mathbb{K} \subset \mathbb{T}$.*

THEOREM 2. *If β_1 and α are simultaneously triangularizable matrices then the singularities of Equation* (58) *are confined. More explicitly, if for a given step m the conditions Equations* (79), (80) *and* (86) *are satisfied then*

$$\beta_{m+1}, \beta_{m+2} \in \mathbb{L}, \quad \beta_{m+3} \in \mathbb{K}, \quad \beta_{m+4} \in \mathbb{T}, \quad \det \beta_{m+4} = O(1), \quad \epsilon \to 0.$$

Proof. From Equations (80) and (86) we conclude that $\beta_m \in \mathbb{K}$ and consequently $\beta_m^{-1} \in \mathbb{L}$. Taking into account this fact, Equation (58) implies that $\beta_{m+1} \in \mathbb{L}$. Therefore, $\beta_{m+1}^{-1} \in \mathbb{K}$ and Equation (58), as $\beta_{m+1} \in \mathbb{L}$, give $\beta_{m+2} \in \mathbb{L}$ and consequently $\beta_{m+2}^{-1} \in \mathbb{K}$. Iterating Equation (58) we get

$$\beta_{m+3} = \beta_m - (m+1)\beta_{m+1}^{-1} + (m+2)\beta_{m+2}^{-1}. \tag{93}$$

Using the just derived facts, $\beta_{m+1}^{-1}, \beta_{m+2}^{-1} \in \mathbb{K}$, and that $\beta_m \in \mathbb{K}$, we deduce $\beta_{m+3} \in \mathbb{K}$ which implies $\beta_{m+3}^{-1} \in \mathbb{L}$. Finally, Equation (58) gives β_{m+4} as

$$\beta_{m+4} = (m+3)\beta_{m+3}^{-1} - \beta_{m+2} - \beta_{m+3} - \alpha. \tag{94}$$

We conclude that there are only two possibilities:

1. $\beta_{m+4} = O(1)$ for $\epsilon \to 0$, or
2. $\beta_{m+4} \in \mathbb{L}$.

Let us consider both possibilities separately.

1. Recalling that the diagonal part has singularity confinement, see Proposition 11, in the first case we see that $\det \beta_{m+4} = O(1)$ when $\epsilon \to 0$, as desired.
2. In this second case we write β_{m+4} as

$$\beta_{m+4} = \beta_{m+3}^{-1} A + O(1), \quad \epsilon \to 0, \quad A := (m+3)\mathbb{I} - \beta_{m+3}\beta_{m+2}. \tag{95}$$

Observe that the repeated use of Equation (58) leads to the following expressions:

$$\begin{aligned} A &= \mathbb{I} + \big[(m+1)\beta_{m+1}^{-1} - \beta_m\big]\beta_{m+2} \\ &= k + \mathbb{I} - \big[(m+1)\beta_{m+1}^{-1} - \beta_m\big]\beta_{m+1} \\ &= k - m\mathbb{I} + \beta_m\beta_{m+1} \\ &= k - \beta_m(\beta_m + \beta_{m-1} + \alpha), \end{aligned}$$

where

$$k := \big[(m+1)\beta_{m+1}^{-1} - \beta_m\big]\big[(m+1)\beta_{m+1}^{-1} - \beta_m - \alpha\big].$$

From these formulae, as $\beta_{m+1}^{-1}, \beta_m \in \mathbb{K}$ we deduce that $k \in \mathbb{K}$ and also that $\beta_m(\beta_m + \beta_{m-1} + \alpha) \in \mathbb{K}$. Hence, we conclude that $A \in \mathbb{K}$ and from Equation (95) and Proposition 12 we deduce that $\beta_{m+4} = O(1)$ when $\epsilon \to 0$. Consequently, we arrive to a contradiction, and only possibility 1) remains. ∎

Acknowledgments

The authors thanks economical support from the Spanish Ministerio de Ciencia e Innovación, research project FIS2008-00200. GAC acknowledges the support of the grant Universidad Complutense de Madrid. Finally, MM reckons illuminating discussions with Dr. Mattia Cafasso in relation with orthogonality and singularity confinement, and both authors are grateful to Prof. Gabriel Álvarez Galindo for several discussions and for the experimental confirmation, via MATHEMATICA, of the existence of the confinement of singularities in the 2×2 case.

References

1. P. PAINLEVÉ, LEÇONS sur la théorie analytique des equations différentielles (Leçons de Stockholm, delivered in 1895), Hermann, Paris, 1897. Reprinted in *Œvres de Paul Painlevé, vol. I, Éditions du CNRS*, Paris, 1973.
2. R. CONTE (Ed.), *The Painlevé Property, One Century Later*, Springer Verlag, New York, 1999.
3. B. GRAMMATICOS, A. RAMANI, and V. PAPAGEORGIOU, Do integrable mappings have the Painlevé property? *Phys. Rev. Lett.* 67:1825–1828 (1991).
4. B. GRAMMATICOS, A. RAMANI, and J. HIETARINTA, Discrete versions of the Painlevé equations, *Phys. Rev. Lett.* 67:1829–1832 (1991).
5. A. RAMANI, D. TAKAHASHI, B. GRAMMATICOS, and Y. OHTA, The ultimate discretisation of the Painlevé equations, *Physica D* 114:185–196 (1998).
6. J. HIETARINTA and C. VIALLET, Discrete Painlevé I and singularity confinement in projective space, *Chaos, Solitons and Fractals* 11:29–32 (2000).
7. S. LAFORTUNE and A. GORIELY, Singularity confinement and algebraic integrability, *J. Math. Phys.* 45:1191–1208 (2004).
8. G. FREUD, On the coefficients in the recursion formulae of orthogonal polynomials, *Proc. R. Irish Acad.* A76:1–6 (1976).
9. W. VAN ASSCHE, Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials, in *Proceedings of the International Conference on Difference Equations, special Functions and Orthogonal Polynomials*, pp. 687–725, World Scientific, Hackensack, NJ (2007).
10. A. P. MAGNUS, Freud's equations for orthogonal polynomials as discrete Painlevé equations, in *Symmetries and Integrability of Difference Equations* (Canterbury, 1996), London Math. Soc. Lecture Note Ser., 255, pp. 228–243, Cambridge University Press Cambridge, UK (1999).
11. D. DAMANIK, A. PUSHNITSKI, and B. SIMON, The analytic theory of matrix orthogonal polynomials, *Surv. Approx. Theory* 4:1–85 (2008).
12. A. S. FOKAS, A. R. ITS, and A. V. KITAEV, The isomonodromy approach to matrix models in 2D quantum gravity, *Commun. Math. Phys.* 147:395–430 (1992).
13. E. DAEMS and A. B. J. KUIJLAARS, Multiple orthogonal polynomials of mixed type and non-intersecting Brownian motions, *J. Approx. Theory* 146:91–114 (2007).
14. M. G. KREIN, Infinite J-matrices and a matrix moment problem, *Dokl. Akad. Nauk. SSSR* 69:125–128 (1949).

15. M. G. Krein, *Fundamental Aspects of the Representation Theory of Hermitian Operators with Deficiency Index (m,m), AMS Translations, Series 2*, Vol. 97, pp. 75–143, Providence, Rhode Island, 1971.
16. Yu. M. Berezanskii, Expansions in eigenfunctions of self-adjoint operators, Transl. Math. Monographs 17, Amer. Math. Soc. Providence, RI, 1968.
17. J. S. Geronimo, Scattering theory and matrix orthogonal polynomials on the real line, *Circ. Syst. Sig. Proc.* 1:471–495 (1982).
18. A. I. Aptekarev and E. M. Nikishin, The scattering problem for a discrete Sturm–Liouville operator, *Math. USSR-Sb.* 49:325–355 (1984).
19. A. J. Durán and F. J. Grünbaum, Orthogonal matrix polynomials, scalar-type Rodrigues' formulas and Pearson equations, *J. Approx. Theory* 134:267–280 (2005).
20. A. J. Durán and F. J. Grünbaum, Structural formulas for orthogonal matrix polynomials satisfying second order differential equations, I, *Constr. Approx.* 22:255–271 (2005).
21. R. D. Costin, Matrix valued polynomials generated by the scalar-type Rodrigues' formulas, *J. Approx. Theory* 161:693–705 (2009).
22. A. J. Durán, Matrix inner product having a matrix symmetric second order differential operator, *Rocky Mount. J. Math.* 27:585–600 (1997).
23. A. J. Durán and F. J. Grünbaum, Orthogonal matrix polynomials satisfying second order differential equations, *Int. Math. Res. Notices* 10:461–484 (2004).
24. J. Borrego, M. Castro, and A. J. Durán, Orthogonal matrix polynomials satisfying differential equations with recurrence coefficients having non-scalar limits, *Integral Transforms and Special Functions* 1–16 (2011).
25. A. J. Durán and M. D. de la Iglesia, Second order differential operators having several families of orthogonal matrix polynomials as eigenfunctions, *Int. Math. Res. Not.*, Article ID rnn084, 24 p. (2008).
26. M. J. Cantero, L. Moral, and L. Velázquez, Differential properties of matrix orthogonal polynomials, *J. Concrete Appl. Math.* 3:313–334 (2005).

Universidad Complutense de Madrid

(Received June 14, 2011)

CHAPTER III

SINGULARITY CONFINEMENT FOR MATRIX DISCRETE PAINLEVÉ EQUATIONS

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **27** (2014) 2321–2335 doi:10.1088/0951-7715/27/9/2321

Singularity confinement for matrix discrete Painlevé equations

Giovanni A Cassatella-Contra[1], **Manuel Mañas**[1] **and Piergiulio Tempesta**[1,2]

[1] Departamento de Física Teórica II (Métodos Matemáticos de la Física), Facultad de Físicas, Universidad Complutense de Madrid, 28040 Madrid, Spain
[2] Instituto de Ciencias Matemáticas, C/ Nicolás Cabrera, No 13–15, 28049 Madrid, Spain

E-mail: p.tempesta@fis.ucm.es, gaccontra@fis.ucm.es and manuel.manas@ucm.es

Received 6 December 2013, revised 13 July 2014
Accepted for publication 24 July 2014
Published 14 August 2014

Recommended by T Grava

Abstract

We study the analytic properties of a matrix discrete system introduced by Cassatella and Mañas (2012 *Stud. Appl. Math.* **128** 252–74). The singularity confinement for this system is shown to hold generically, i.e. in the whole space of parameters except possibly for algebraic subvarieties. This paves the way to a generalization of Painlevé analysis to discrete matrix models.

Keywords: singularity confinement, discrete integrable systems, noncommutative discrete Painlevé I equation, Schur complements
Mathematics Subject Classification: 46L55, 37K10, 37L60

1. Introduction

Since the discovery of the *Painlevé property* for ordinary differential equations at the end of the 19th century [21], the notion of *integrability* has been related to the local analysis of movable isolated singularities of solutions of dynamical systems [8]. This approach to integrability has opened an alternative perspective compared with the standard algebraic approach *à la Liouville*, based on the existence of a suitable number of functionally independent integrals of motion. Both points of view have been extended to the study of evolution equations on a discrete background.

Integrable discrete systems, for several aspects more fundamental objects than the continuous ones, are ubiquitous in both pure and applied mathematics, and in theoretical physics as well. They possess rich algebro-geometric properties [3,5,9,18,25] and are relevant, for instance, in the regularization of quantum field theories in a lattice and in discrete quantum gravity [10,16].

In particular, the problem of integrability preserving discretizations of partial differential equations has become a very active research area [23], and has been widely investigated with both geometrical and algebraic methods [5, 6, 20, 24].

The approach known as *singularity confinement*, introduced in [13], is the equivalent for discrete systems of the singularity analysis for continuous dynamical systems. It essentially relies on the observation that for integrable discrete models, if a singularity appears in some specific point of the lattice of the independent variable, then it would disappear after making the system evolve via a finite number of iterations. Alternative, related approaches are based on the notion of algebraic entropy [4, 17] or on Nevalinna theory [1, 22]. A large class of difference equations coming from unitary integrals and combinatorics possess the confinement property [2]. However, observe that singularity confinement, in spite of being extremely useful in isolating integrability, might not be a sufficient condition for integrability, as was observed by Hietarinta and Viallet [15].

The purpose of this paper is to start a theoretical study of the singularity confinement property for *matrix integrable systems*. Indeed, we hypothesize that the singularity analysis has the same relevance for matrix systems that it possesses for both discrete and continuous scalar models.

Apart from its intrinsic mathematical interest, the study of matrix discrete dynamical systems can also be related, from an applicative point of view, to the theory of complex networks [19]. Indeed, given a random graph with N vertices, one associates with it the adjacency matrix, which is a $N \times N$ matrix, whose entries a_{ij} represent the number of links associated with the nodes i and j ($i, j = 1, \dots, N$). The discrete time evolution of the topology of the network would provide a difference equation for the adjacency matrix, defining a discrete matrix model.

Hereafter, we shall focus on the singularity confinement of the following discrete matrix equation

$$\beta_{n+1} = n\beta_n^{-1} - \beta_{n-1} - \beta_n - \alpha, \qquad n = 1, 2, \dots \tag{1}$$

where $\beta_n \in \mathbb{C}^{N\times N}$ is a $N \times N$ complex matrix.

Equation (1) can be considered a kind of non-Abelian matrix version of the discrete Painlevé equation (dPI). It was introduced in [7], and soon after studied in [14], and describes the recursion relation for the matrix coefficients of a class of Freud matrix orthogonal polynomials with a quartic potential [11] in the context of the associated Riemann–Hilbert problem. In that paper we also proved the singularity confinement in a simple situation, when the initial data are triangular matrices up to similarity transformations. The aim of this paper is to extend this result to the general case. This extension relies heavily on the use of Schur complements, which appear often in the analysis of non-Abelian systems, see [12]. It should also be remarked that this proof required deeper understanding and study than in the triangularizable situation. The difficulty mainly resides in the analysis of the genericness of the result given in theorem 2.

1.1. Preliminary discussion

Let us present here the simplest case of singularity analysis for the matrix model (1), which parallels the results for the standard discrete Painlevé I equation. We assume that β_{m-1} do not depend on ϵ and that

$$\beta_m = \beta_{m,1}\epsilon + \beta_{m,2}\epsilon^2 + O(\epsilon^3), \qquad \epsilon \to 0, \tag{2}$$

with $\det \beta_{m,1} \neq 0$. Observe that we are assuming the leading term for β_m is proportional to ϵ, we say that we have a 'zero'. Note also that the leading term coefficient is required, in this example, to be invertible. This is the only possibility in the scalar case $N = 1$, but as we will discuss later the non-Abelian scenario $N \geqslant 2$ implies a richer situation. Thus, as this approach will hold hereon, we assume that at some integer m of the lattice a zero appears, while for the previous one, $m-1$, neither a zero nor singularity shows up.

If we introduce condition (2) into (1), we have that

$$\beta_{m+1} = m\beta_{m,1}^{-1}\epsilon^{-1} + \beta_{m+1,0} + \beta_{m+1,1}\epsilon + \beta_{m+1,2}\epsilon^2 + O(\epsilon^3), \tag{3}$$

where

$$\begin{aligned}
\beta_{m+1,0} &= -m\beta_{m,1}^{-1}\beta_{m,2}\beta_{m,1}^{-1} - \beta_{m-1} - \alpha,\\
\beta_{m+1,1} &= m\beta_{m,1}^{-1}(\beta_{m,2}\beta_{m,1}^{-1}\beta_{m,2} - \beta_{m,3})\beta_{m,1}^{-1} - \beta_{m,1},\\
\beta_{m+1,2} &= m\big(\beta_{m,2}\beta_{m,1}^{-1}(\beta_{m,3} - \beta_{m,2}\beta_{m,1}^{-1}\beta_{m,2}) + \beta_{m,3}\beta_{m,1}^{-1}\beta_{m,2} - \beta_{m,4}\big)\beta_{m,1}^{-2} - \beta_{m,2}.
\end{aligned}$$

We observe that a leading term in ϵ^{-1} appeared in the asymptotic expansion. This 'pole singularity' will survive still for another step in the sequence

$$\beta_{m+2} = -m\beta_{m,1}^{-1}\epsilon^{-1} + \beta_{m+2,0} + \beta_{m+2,1}\epsilon + \beta_{m+2,2}\epsilon^2 + O(\epsilon^3), \tag{4}$$

where

$$\begin{aligned}
\beta_{m+2,0} &= m\beta_{m,1}^{-1}\beta_{m,2}\beta_{m,1}^{-1} + \beta_{m-1},\\
\beta_{m+2,1} &= \frac{(m+1)}{m}\beta_{m,1} - m\beta_{m,1}^{-1}\beta_{m,2}\beta_{m,1}^{-1}\beta_{m,2}\beta_{m,1}^{-1} + m\beta_{m,1}^{-1}\beta_{m,3}\beta_{m,1}^{-1},\\
\beta_{m+2,2} &= \frac{(m+1)}{m}\beta_{m,2} + \frac{(m+1)}{m^2}\beta_{m,1}(\beta_{m-1}+\alpha)\beta_{m,1}\\
&\quad + m\beta_{m,2}\beta_{m,1}^{-1}(\beta_{m,2}\beta_{m,1}^{-1}\beta_{m,2}\beta_{m,1}^{-2} - \beta_{m,3}\beta_{m,1}^{-2}) - m\beta_{m,3}\beta_{m,1}^{-1}\beta_{m,2}\beta_{m,1}^{-2} + m\beta_{m,4}\beta_{m,1}^{-2}.
\end{aligned}$$

We easily check that in the third step the leading term is proportional to ϵ, this 'zero' appears again

$$\beta_{m+3} = \frac{-(m+3)}{m}\beta_{m,1}\epsilon + \beta_{m+3,2}\epsilon^2 + O(\epsilon^3), \tag{5}$$

where

$$\beta_{m+3,2} := -\frac{(m+3)}{m}\beta_{m,2} - \frac{(2m+3)}{m^2}\beta_{m,1}\beta_{m-1}\beta_{m,1} - \frac{(m+1)}{m^2}\beta_{m,1}\alpha\beta_{m,1}.$$

Finally, if we substitute (4) and (5) into (1) we obtain no singularity at all:

$$\beta_{m+4} = \frac{m}{(m+3)}\beta_{m-1} - \frac{2}{(m+3)}\alpha + O(\epsilon).$$

Observe that $\beta_{m+3} = O(\epsilon)$, $\beta_{m+4} = O(1)$ and $\det \beta_{m+4} = O(1)$ for $\epsilon \to 0$. Thus, unless

$$\det(m\beta_{m-1} - 2\alpha) = 0, \tag{6}$$

we obtain singularities in the step just after the appearance of a *zero* in β_m, with the poles appearing in the sites $m+1$, $m+2$. Then we have a *zero* for $m+3$ while we recover the standard behaviour for $m+4$. A crucial point is that this singularity confinement holds whenever (6) is *not* satisfied. This observation motivates the definitions proposed in the following discussion.

Definition 1. *Whenever the singularity confinement property is satisfied in the whole space $\mathcal{S}$ of parameters except possibly for a set of algebraic subvarieties $\mathcal{W}_i \in \mathcal{S}$, $i = 1, , , j \in \mathbb{N}$, we shall say that the property is satisfied generically.*

In this case we will speak about the genericness of the singularity confinement.

Definition 2. *We shall define the confinement time as the minimum number $l \in \mathbb{N}$ of iterations or steps in the lattice, after the appearance of a zero, necessary to recover the form without poles or zeros.*

Thus, in the above case we have generically a singularity confinement with a confinement time $l = 4$.

A simple but fundamental observation for the sequel of the paper is the following one.

Lemma 1. *The matrix system (1) is invariant under similarity transformations.*

Proof. Observe that

$$M\beta_{n+1}M^{-1} = nM\beta_n^{-1}M^{-1} - M\beta_{n-1}M^{-1} - M\beta_n M^{-1} - M\alpha M^{-1}.$$

Therefore, we obtain

$$\phi_{n+1} = n\phi_n^{-1} - \phi_{n-1} - \phi_n - \delta,$$

where $\phi_n := M\beta_n M^{-1}$ and $\delta := M\alpha M^{-1}$. □

1.2. Main result

The ideas developed within the previous example will be used in the subsequent considerations to study the confinement of the singularities of the matrix dPI model (1). In this noncommutative scenario we must be careful when we talk about zeroes and singularities associated with asymptotic expansions. For the example discussed above it was just as in the Abelian case with $N = 1$ as we assumed that the leading term coefficients of the zero was an invertible matrix. In general this is just not the case and we need to consider the rank, $\operatorname{rank}(\beta_{m,0})$, of the matrix coefficient of the leading term of β_m.

As before let us suppose that for some integer m of the lattice a zero appears, while for $(m-1)$ neither a zero nor singularity shows up. But now we must carefully explain what we mean by a zero. We shall assume that β_{m-1} do not depend on ϵ and that

$$\beta_m = \beta_{m,0} + \beta_{m,1}\epsilon + O(\epsilon^2), \qquad \det\beta_m = O(\epsilon^r), \quad \epsilon \to 0, \tag{7}$$

where $\beta_{m,i} \in \mathbb{C}^{N\times N}$ and $r \in \{1, \dots, N\}$. Consequently, we can distinguish two cases.

- $r = N$. This is the maximal rank case discussed above; for it we have that

$$\beta_{m,0} = 0, \qquad \det\beta_{m,1} \neq 0.$$

As we have already seen it presents singularity confinement generically.

- $r \leqslant N - 1$. For the non-maximal rank case we instead have

$$\begin{aligned} &\operatorname{rank}(\beta_{m,0}) = N - r,\\ &\det\beta_m = O(\epsilon^r), \qquad \epsilon \to 0. \end{aligned} \tag{8}$$

As will be proven later, using the invariance under a similarity transformation, one can assume that the matrices β will have the form expressed by equation (13). So said, we can state the main result of the paper as follows.

Theorem 1. *If β_{m-1} do not depend on ϵ and β_m is of the form (7), and the following conditions for $\epsilon \to 0$ are satisfied*

$$\det\beta_{m+1} = O(\epsilon^{-r}), \tag{9}$$

$$\det\beta_{m+2} = O(\epsilon^{-r}), \tag{10}$$

$$\det\beta_{m+3} = O(\epsilon^{r}), \tag{11}$$

$$\det\beta_{m+4} = O(1), \tag{12}$$

then, there is singularity confinement for the dPI model (1) with confinement time $l = 4$.

It is important to remark that conditions (9)–(12) can be proven to hold generically, that is the content of theorem (2). Therefore, we can state that our system generically has the singularity confinement property.

2. $N \times N$ matrix asymptotic expansions and singularity confinement

In this section we will consider the set of matrix asymptotic expansions

$$\mathcal{A} = \mathbb{C}^{N\times N}(\!(\epsilon)\!) := \left\{M_0 + M_1\epsilon + O(\epsilon^2),\ \epsilon \to 0,\ M_i \in \mathbb{C}^{N\times N}\right\}.$$

This set is a ring with identity, given by the matrix $\mathbb{I}_N$. For each possible rank $r \in \{1, \dots, N-1\}$ we will use the block notation

$$M := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad A \in \mathbb{C}^{r\times r}, B \in \mathbb{C}^{r\times(N-r)}, C \in \mathbb{C}^{(N-r)\times r}, D \in \mathbb{C}^{(N-r)\times(N-r)}.$$

We also introduce two subalgebras of the algebra $\mathbb{C}^{N\times N}$

$$\mathfrak{K} := \left\{K = \begin{pmatrix} 0 & 0 \\ K_{21} & K_{22} \end{pmatrix}, K_{21} \in \mathbb{C}^{(N-r)\times r}, K_{22} \in \mathbb{C}^{(N-r)\times(N-r)}\right\},$$

$$\mathfrak{L} := \{L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & 0 \end{pmatrix}, L_{11} \in \mathbb{C}^{r\times r}, L_{21} \in \mathbb{C}^{(N-r)\times r}\Big\},$$

and the related subsets of matrix asymptotic expansions

$$\mathcal{A}_{\mathfrak{K}} := \{K \in \mathcal{A}, K|_{\epsilon=0} \in \mathfrak{K}\}, \qquad \mathcal{A}_{\mathfrak{L}} := \{L \in \mathcal{A}, L|_{\epsilon=0} \in \mathfrak{L}\},$$

which satisfy several important properties.

Proposition 1. *The following statements hold.*

(1) Both $\mathcal{A}_{\mathfrak{K}}$ and $\mathcal{A}_{\mathfrak{L}}$ are subrings without identity of the ring $\mathcal{A}$.
(2) For $K \in \mathcal{A}_{\mathfrak{K}}$ such that $\det K = O(\epsilon^r)$, $\epsilon \to 0$, then $K^{-1} \in \epsilon^{-1}\mathcal{A}_{\mathfrak{L}}$, and reciprocally if $L \in \epsilon^{-1}\mathcal{A}_{\mathfrak{L}}$ with $\det L = O(\epsilon^{-r})$, $\epsilon \to 0$, then $L^{-1} \in \mathcal{A}_{\mathfrak{K}}$.
(3) If $K \in \mathcal{A}_{\mathfrak{K}}$, that is $K = \begin{pmatrix} 0 & 0 \\ C_0 & D_0 \end{pmatrix} + \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\epsilon + O(\epsilon^2)$ then

$$\det K = \epsilon^r \det \begin{pmatrix} A_1 & B_1 \\ C_0 & D_0 \end{pmatrix} + O(\epsilon^{r+1}), \qquad \epsilon \to 0.$$

(4) If $L \in \epsilon^{-1}\mathcal{A}_{\mathfrak{L}}$, that is $L = \begin{pmatrix} A_0 & 0 \\ C_0 & 0 \end{pmatrix}\epsilon^{-1} + \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} + O(\epsilon)$ then

$$\det L = \epsilon^{-r} \det \begin{pmatrix} A_0 & B_1 \\ C_0 & D_1 \end{pmatrix} + O(\epsilon^{-r+1}), \qquad \epsilon \to 0.$$

(5) The subrings $\mathcal{A}_{\mathfrak{K}}$ and $\mathcal{A}_{\mathfrak{L}}$ are right and left ideals of $\mathcal{A}$, respectively, i.e. $\mathcal{A}_{\mathfrak{K}} \cdot \mathcal{A} \subset \mathcal{A}_{\mathfrak{K}}$ and $\mathcal{A} \cdot \mathcal{A}_{\mathfrak{L}} \subset \mathcal{A}_{\mathfrak{L}}$.
(6) The following inclusion holds: $\epsilon^{-1}\mathcal{A}_{\mathfrak{L}} \cdot \mathcal{A}_{\mathfrak{K}} \subset \mathcal{A}$.

The proof of the previous statements is direct and left to the reader.

To study the singularity confinement of the matrix equation (1) when β_n satisfies conditions (8), we shall use expressions (7), having applied a similarity transformation to β such that

$\beta_{m,0} \in \mathfrak{K}$, $\beta_m \in \mathcal{A}_{\mathfrak{K}}$. In other words

$$\beta_{m,0} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \beta_{m,0;r+1,1} & \beta_{m,0;r+1,2} & \cdots & \beta_{m,0;r+1,r+1} & \beta_{m,0;r+1,r+2} & \cdots & \beta_{m,0;r+1,N} \\ \beta_{m,0;r+2,1} & \beta_{m,0;r+2,2} & \cdots & \beta_{m,0;r+2,r+1} & \beta_{m,0;r+2,r+2} & \cdots & \beta_{m,0;r+2,N} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \beta_{m,0;N,1} & \beta_{m,0;N,2} & \cdots & \beta_{m,0;N,r+1} & \beta_{m,0;N,r+2} & \cdots & \beta_{m,0;N,N} \end{pmatrix}, \tag{13}$$

where $m \geqslant 2$, and all the entries that are above the r+1-th row of β_m are zero. Notice that β_{m-1} and β_m belong to the rings $\mathcal{A}$ and $\mathcal{A}_{\mathfrak{K}}$, respectively.

2.1. Proof of the theorem 1

Proof. As $\beta_{m,0} \in \mathfrak{K}$, i.e. $\beta_m \in \mathcal{A}_{\mathfrak{K}}$, and by hypothesis $\det \beta_m = O(\epsilon^r)$, $\epsilon \to 0$, proposition 1 implies

$$\beta_m^{-1} = (\beta_m^{-1})_{-1}\epsilon^{-1} + (\beta_m^{-1})_0 + O(\epsilon), \quad \epsilon \to 0, \quad (\beta_m^{-1})_{-1} \in \mathfrak{L}. \tag{14}$$

If we replace equations (7) and (13) into equation (1) we deduce

$$\beta_{m+1} = m\beta_m^{-1} + O(1), \qquad \epsilon \to 0.$$

Using the relations (14), (7) and (13), this expression is reduced to

$$\beta_{m+1} = m(\beta_m^{-1})_{-1}\epsilon^{-1} + O(1), \qquad \epsilon \to 0. \tag{15}$$

Since $(\beta_m^{-1})_{-1} \in \mathfrak{L}$, from (15) we conclude that $\beta_{m+1} \in \epsilon^{-1}\mathcal{A}_{\mathfrak{L}}$, showing a simple pole singularity. Due to the fact that by hypothesis equation (9) holds, proposition 1 implies

$$\beta_{m+1}^{-1} \in \mathcal{A}_{\mathfrak{K}}. \tag{16}$$

Then we deduce

$$\beta_{m+2} = -m(\beta_m^{-1})_{-1}\epsilon^{-1} + O(1), \quad \epsilon \to 0, \quad \beta_{m+2} \in \mathcal{A}_{\mathfrak{L}}.$$

As before, using condition (10), proposition 1 gives

$$\beta_{m+2}^{-1} \in \mathcal{A}_{\mathfrak{K}}.$$

Now,

$$\beta_{m+3} = \beta_m - (m+1)\beta_{m+1}^{-1} + (m+2)\beta_{m+2}^{-1}, \tag{17}$$

where in the rhs we have used twice equation (1) to write β_{m+2} as a function of β_{m+1} and β_m. As we have proven that $\beta_m, \beta_{m+1}^{-1}, \beta_{m+2}^{-1} \in \mathcal{A}_{\mathfrak{K}}$, we deduce that

$$\beta_{m+3} \in \mathcal{A}_{\mathfrak{K}}.$$

As a consequence of equation (11) and proposition 1, we obtain

$$\beta_{m+3}^{-1} \in \epsilon^{-1}\mathcal{A}_{\mathfrak{L}}. \tag{18}$$

Our matrix discrete Painlevé equation (1) gives

$$\beta_{m+4} = (m+3)\beta_{m+3}^{-1} - \beta_{m+2} - \beta_{m+3} - \alpha,$$

which implies

$$\beta_{m+4} = \beta_{m+3}^{-1} A + O(1), \qquad \epsilon \to 0, \qquad A := (m+3)\mathbb{I}_N - \beta_{m+3}\beta_{m+2}, \tag{19}$$

where we have taken into account that β_{m+3} and α are $O(1)$. We study the matrix A, by applying equation (1) once. We obtain

$$\begin{aligned} A &= \mathbb{I}_N + [(m+1)\beta_{m+1}^{-1} - \beta_m]\beta_{m+2} \\ &= [(m+1)\beta_{m+1}^{-1} - \beta_m][(m+1)\beta_{m+1}^{-1} - \beta_m - \alpha] - m\mathbb{I}_N + \beta_m\beta_{m+1} \\ &= [(m+1)\beta_{m+1}^{-1} - \beta_m][(m+1)\beta_{m+1}^{-1} - \beta_m - \alpha] - \beta_m(\beta_m + \beta_{m-1} + \alpha). \end{aligned} \tag{20}$$

Now, recalling that $\beta_{m-1} = O(1)$, $\beta_m, \beta_{m+1}^{-1} \in \mathcal{A}_{\mathfrak{K}}$, and by virtue of proposition 1 we conclude that

$$A \in \mathcal{A}_{\mathfrak{K}}. \tag{21}$$

Finally, from equations (18), (19) and (21) we deduce that

$$\beta_{m+4} \in \mathcal{A}.$$

By taking into account that $\det \beta_{m+4} = O(1)$, we have proven that the singularity has disappeared. Thus, the singularity confinement is ensured with a confinement time $l = 4$. □

In order to show the genericness of conditions (9)–(12) we use the block notation

$$\beta_{m-1} = \begin{pmatrix} A_{m-1} & B_{m-1} \\ C_{m-1} & D_{m-1} \end{pmatrix}, \qquad \alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}.$$

and consider the expansion

$$\beta_m = \begin{pmatrix} 0 & 0 \\ C_{m,0} & D_{m,0} \end{pmatrix} + \sum_{i=1}^{\infty} \begin{pmatrix} A_{m,i} & B_{m,i} \\ C_{m,i} & D_{m,i} \end{pmatrix} \epsilon^i.$$

Definition 3. *We introduce*

$$\begin{aligned} Z_1 &:= D_{m+1,0} + D_{m,0}^{-1} C_{m,0} B_{m+1,0}, \\ Z_2 &:= D_{m+2,0} + D_{m,0}^{-1} C_{m,0} B_{m+2,0}, \\ Z_3 &:= D_{m+3,0}. \end{aligned}$$

The genericness of the singularity confinement can be stated as follows.

Theorem 2. *(1) If* $\det D_{m,0} \neq 0$, *for* $\epsilon \to 0$ *we have*

$$\det \beta_{m+1} = O(\epsilon^{-r}) \Leftrightarrow \det(Z_1) \neq 0.$$

(2) If $\det D_{m,0} \neq 0$, $\det Z_1 \neq 0$, *we have that for* $\epsilon \to 0$

$$\det \beta_{m+2} = O(\epsilon^{-r}) \Leftrightarrow \det(Z_2) \neq 0.$$

(3) If $\det D_{m,0} \neq 0$, $\det Z_1 \neq 0$ *and* $\det Z_2 \neq 0$, *we have that for* $\epsilon \to 0$

$$\det \beta_{m+3} = O(\epsilon^{r}) \Leftrightarrow \det Z_3 \neq 0.$$

(4) If $\det D_{m,0} \neq 0$, $\det Z_1 \neq 0$, $\det Z_2 \neq 0$ *and* $\det Z_3 \neq 0$ *we have that*

$$\det \beta_{m+4} = O(1), \qquad \epsilon \to 0,$$

generically.

Proof. See appendix B. □

The matrices Z_1, Z_2 and Z_3 can be expressed in terms of initial conditions as follows.

Proposition 2. *The following expressions in terms of initial conditions hold:*

$$
\begin{aligned}
Z_1 &= mD_{m,0}^{-1} - D_{m-1} - D_{m,0} - \alpha_{22} - D_{m,0}^{-1}C_{m,0}(B_{m-1}+\alpha_{12}),\\
Z_2 &= (m+1)(mD_{m,0}^{-1} - D_{m,0}^{-1}C_{m,0}(B_{m-1}+\alpha_{12}) - D_{m-1} - D_{m,0} - \alpha_{22})^{-1}\\
&\quad + D_{m,0}^{-1}C_{m,0}B_{m-1} - mD_{m,0}^{-1} + D_{m-1},\\
Z_3 &= D_{m,0} - (m+1)Z_1^{-1} + (m+2)Z_2^{-1}.
\end{aligned}
$$

Proof. Is a byproduct of the proof of theorem 2. □

Appendix A. Schur complements

To show the genericness of the confinement phenomenon in the non- Abelian scenario it is very convenient to introduce Schur complements.

Definition 4. *Given M in the block form as in (13), the Schur complements with respect to D (if $\det D \neq 0$), and to A (if $\det A \neq 0$) are defined to be*

$$S_D(M) := A - BD^{-1}C, \qquad S_A(M) := D - CA^{-1}B,$$

respectively.

In terms of the Schur complements we have the following well-known expressions for the inverse matrices

$$
M^{-1} = \begin{cases}
\begin{pmatrix} S_D(M)^{-1} & -S_D(M)^{-1}BD^{-1} \\ -D^{-1}CS_D(M)^{-1} & D^{-1}(\mathbb{I}_{N-r} + CS_D(M)^{-1}BD^{-1}) \end{pmatrix}, & \text{for } \det D, \det S_D(M) \neq 0,\\
\begin{pmatrix} A^{-1} + A^{-1}BS_A(M)^{-1}CA^{-1} & -A^{-1}BS_A(M)^{-1} \\ -S_A(M)^{-1}CA^{-1} & S_A(M)^{-1} \end{pmatrix}, & \text{for } \det A, \det S_A(M) \neq 0,\\
\begin{pmatrix} S_D(M)^{-1} & -S_D(M)^{-1}BD^{-1} \\ -D^{-1}CS_D(M)^{-1} & S_A(M)^{-1} \end{pmatrix}, & \text{for } \det A, \det D, \det S_D(M),\\
& \det S_A(M) \neq 0,
\end{cases}
\tag{22}
$$

and for the determinant of M

$$
\begin{aligned}
\det M &= \det A \quad \det S_A(M)\\
&= \det D \quad \det S_D(M).
\end{aligned}
\tag{23}
$$

Now, if $K = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ C_0 & D_0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right)\epsilon + O(\epsilon^2) \in \mathcal{A}_{\mathfrak{K}}$ then we can write the Schur complements in the form

$$
\begin{aligned}
S_D(K) &= A - BD^{-1}C =: S_D(K)_1\epsilon + S_D(K)_2\epsilon^2 + O(\epsilon^3), && \epsilon \to 0,\\
S_A(K) &= D - CA^{-1}B =: S_A(K)_0 + S_A(K)_1\epsilon + O(\epsilon^2), && \epsilon \to 0,
\end{aligned}
\tag{24}
$$

where

$$
\begin{aligned}
S_D(K)_1 &= A_1 - B_1 D_0^{-1} C_0,\\
S_D(K)_2 &= A_2 - B_1 D_0^{-1} C_1 - B_2 D_0^{-1} C_0 + B_1 D_0^{-1} D_1 D_0^{-1} C_0,\\
S_D(K)_3 &= A_3 - B_1 D_0^{-1} C_2 + (B_1 D_0^{-1} D_1 D_0^{-1} - B_2 D_0^{-1}) C_1\\
&\quad + B_1 (D_0^{-1} D_2 D_0^{-1} - D_0^{-1} D_1 D_0^{-1} D_1 D_0^{-1}) C_0 + B_2 D_0^{-1} D_1 D_0^{-1} C_0 - B_3 D_0^{-1} C_0,\\
S_D(K)_4 &= A_4 - B_1 D_0^{-1} C_3 + B_1 D_0^{-1} D_1 D_0^{-1} C_2 - B_1 D_0^{-1} (D_1 D_0^{-1} D_1 D_0^{-1} - D_2 D_0^{-1}) C_1\\
&\quad - B_1 D_0^{-1} D_2 D_0^{-1} D_1 D_0^{-1} C_0 + B_1 D_0^{-1} D_1 (D_0^{-1} D_1 D_0^{-1} D_1 D_0^{-1} - D_0^{-1} D_2 D_0^{-1}) C_0\\
&\quad + B_1 D_0^{-1} D_3 D_0^{-1} C_0 - B_2 D_0^{-1} C_2 + B_2 D_0^{-1} D_1 D_0^{-1} C_1\\
&\quad - B_2 D_0^{-1} (D_1 D_0^{-1} D_1 D_0^{-1} - D_2 D_0^{-1}) C_0 - B_3 D_0^{-1} (C_1 - D_1 D_0^{-1} C_0) - B_4 D_0^{-1} C_0,\\
S_A(K)_0 &= D_0 - C_0 A_1^{-1} B_1,\\
S_A(K)_1 &= D_1 - C_0 A_1^{-1} B_2 - C_1 A_1^{-1} B_1 + C_0 A_1^{-1} A_2 A_1^{-1} B_1.
\end{aligned}
$$

For the determinant $\det M$ we just take into account equations (23) and (24) to obtain

$$
\begin{aligned}
\det K &= \epsilon^r \det(A_1 - B_1 D_0^{-1} C_0 + O(\epsilon)) \det(D_0 + O(\epsilon))\\
&= \det(A_1 - B_1 D_0^{-1} C_0) \det(D_0) \epsilon^r + O(\epsilon^{r+1}).
\end{aligned}
$$

Appendix B. Proof of theorem 2

Lemma 2. *(1) Assuming that* $\det D_{m,0} \neq 0$ *the following asymptotic holds.*

$$
\det \beta_{m+1} = \epsilon^{-r} \begin{vmatrix} m S_D(\beta_m)_1^{-1} & -m S_D(\beta_m)_1^{-1} B_{m,1} D_{m,0}^{-1} - B_{m-1} - \alpha_{12} \\ -m D_{m,0}^{-1} C_{m,0} S_D(\beta_m)_1^{-1} & m D_{m,0}^{-1} + m D_{m,0}^{-1} C_{m,0} S_D(\beta_m)_1^{-1} B_{m,1} D_{m,0}^{-1} \\ & -D_{m-1} - D_{m,0} - \alpha_{22} \end{vmatrix}
+ O(\epsilon^{-r+1})
$$

for $\epsilon \to 0$, *where* $S_D(\beta_m)_1 := A_{m,1} - B_{m,1} D_{m,0}^{-1} C_{m,0} \in \mathbb{C}^{r\times r}$.

Proof. From equation (7) we know that

$$
\det \begin{pmatrix} A_{m,1} & B_{m,1} \\ C_{m,0} & D_{m,0} \end{pmatrix} \neq 0,
$$

hence $S_D(\beta_m)_1$ is invertible. Then, from (22) and (24) we deduce

$$
\begin{aligned}
\beta_m^{-1} &= \begin{pmatrix} (\beta_m^{-1})_{11,-1} & 0 \\ (\beta_m^{-1})_{21,-1} & 0 \end{pmatrix} \epsilon^{-1} + \begin{pmatrix} (\beta_m^{-1})_{11,0} & (\beta_m^{-1})_{12,0} \\ (\beta_m^{-1})_{21,0} & (\beta_m^{-1})_{22,0} \end{pmatrix}\\
&\quad + \begin{pmatrix} (\beta_m^{-1})_{11,1} & (\beta_m^{-1})_{12,1} \\ (\beta_m^{-1})_{21,1} & (\beta_m^{-1})_{22,1} \end{pmatrix} \epsilon + O(\epsilon^2), \quad \epsilon \to 0,
\end{aligned}
$$

where the pole coefficients are

$$
(\beta_m^{-1})_{11,-1} := S_D(\beta_m)_1^{-1}, \qquad (\beta_m^{-1})_{21,-1} := -D_{m,0}^{-1} C_{m,0} S_D(\beta_m)_1^{-1}, \tag{25}
$$

while the regular part coefficients are

$$
\begin{aligned}
(\beta_m^{-1})_{11,0} &:= -S_D(\beta_m)_1^{-1} S_D(\beta_m)_2 S_D(\beta_m)_1^{-1},\\
(\beta_m^{-1})_{12,0} &:= -S_D(\beta_m)_1^{-1} B_{m,1} D_{m,0}^{-1},\\
(\beta_m^{-1})_{21,0} &:= D_{m,0}^{-1}(C_{m,0} S_D(\beta_m)_1^{-1} S_D(\beta_m)_2 S_D(\beta_m)_1^{-1} - (C_{m,1} - D_{m,1} D_{m,0}^{-1} C_{m,0}) S_D(\beta_m)_1^{-1}),\\
(\beta_m^{-1})_{22,0} &:= D_{m,0}^{-1}\big(\mathbb{I}_{N-r} + C_{m,0} S_D(\beta_m)_1^{-1} B_{m,1} D_{m,0}^{-1}\big),
\end{aligned}
$$

$$
\begin{aligned}
(\beta_m^{-1})_{11,1} &:= S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1} - S_D(\beta_m)_1^{-1}S_D(\beta_m)_3S_D(\beta_m)_1^{-1},\\
(\beta_m^{-1})_{12,1} &:= \big(S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1}B_{m,1} - S_D(\beta_m)_1^{-1}(B_{m,2} - B_{m,1}D_{m,0}^{-1}D_{m,1})\big)D_{m,0}^{-1},\\
(\beta_m^{-1})_{21,1} &:= -D_{m,0}^{-1}\big(C_{m,0}[S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1}\\
&\quad - S_D(\beta_m)_1^{-1}S_D(\beta_m)_3S_D(\beta_m)_1^{-1}]\\
&\quad - (C_{m,1} - D_{m,1}D_{m,0}^{-1}C_{m,0})S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1} +\\
&\quad - \big((D_{m,1}D_{m,0}^{-1}D_{m,1} - D_{m,2})D_{m,0}^{-1}C_{m,0} + C_{m,2} - D_{m,1}D_{m,0}^{-1}C_{m,1}\big)S_D(\beta_m)_1^{-1}\big),\\
(\beta_m^{-1})_{22,1} &:= D_{m,0}^{-1}\big(-D_{m,1}+(C_{m,1} - D_{m,1}D_{m,0}^{-1}C_{m,0}-C_{m,0}S_D(\beta_m)_1^{-1}S_D(\beta_m)_2)S_D(\beta_m)_1^{-1}B_{m,1}\\
&\quad + C_{m,0}S_D(\beta_m)_1^{-1}(B_{m,2} - B_{m,1}D_{m,0}^{-1}D_{m,1})\big)D_{m,0}^{-1},\\
(\beta_m^{-1})_{11,2} &:= S_D(\beta_m)_1^{-1}S_D(\beta_m)_3S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1}\\
&\quad - S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1}(S_D(\beta_m)_2S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1}\\
&\quad - S_D(\beta_m)_3S_D(\beta_m)_1^{-1}) - S_D(\beta_m)_1^{-1}S_D(\beta_m)_4S_D(\beta_m)_1^{-1},\\
(\beta_m^{-1})_{12,2} &:= S_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1}(D_{m,2}D_{m,0}^{-1} - D_{m,1}D_{m,0}^{-1}D_{m,1}D_{m,0}^{-1}) + S_D(\beta_m)_1^{-1}(B_{m,2}\\
&\quad - S_D(\beta_m)_2S_D(\beta_m)_1^{-1}B_{m,1})D_{m,0}^{-1}D_{m,1}D_{m,0}^{-1}\\
&\quad - S_D(\beta_m)_1^{-1}(B_{m,3} - S_D(\beta_m)_2S_D(\beta_m)_1^{-1}B_{m,2}\\
&\quad + S_D(\beta_m)_2S_D(\beta_m)_1^{-1}S_D(\beta_m)_2S_D(\beta_m)_1^{-1}B_{m,1} - S_D(\beta_m)_3S_D(\beta_m)_1^{-1}B_{m,1})D_{m,0}^{-1}.
\end{aligned}
$$

Finally, from equation (1) we deduce

$$
\begin{aligned}
\beta_{m+1} = &\begin{pmatrix} mS_D(\beta_m)_1^{-1} & 0\\ -mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1} & 0\end{pmatrix}\epsilon^{-1} + \begin{pmatrix} A_{m+1,0} & B_{m+1,0}\\ C_{m+1,0} & D_{m+1,0}\end{pmatrix}\\
&+ \begin{pmatrix} A_{m+1,1} & B_{m+1,1}\\ C_{m+1,1} & D_{m+1,1}\end{pmatrix}\epsilon + O(\epsilon^2),\ \epsilon\to 0, \qquad (26)
\end{aligned}
$$

where

$$A_{m+1,0} := m(\beta_m^{-1})_{11,0} - A_{m-1} - \alpha_{11}, \quad B_{m+1,0} := m(\beta_m^{-1})_{12,0} - B_{m-1} - \alpha_{12}, \qquad (27)$$

$$C_{m+1,0} := m(\beta_m^{-1})_{21,0} - C_{m-1} - C_{m,0} - \alpha_{21}, \quad D_{m+1,0} := m(\beta_m^{-1})_{22,0} - D_{m-1} - D_{m,0} - \alpha_{22}, \qquad (28)$$

$$A_{m+1,1} := m(\beta_m^{-1})_{11,1} - A_{m-1} - A_{m,1}, \quad B_{m+1,1} := m(\beta_m^{-1})_{12,1} - B_{m-1} - B_{m,1}, \qquad (29)$$

$$C_{m+1,1} := m(\beta_m^{-1})_{21,1} - C_{m,1}, \quad D_{m+1,1} := m(\beta_m^{-1})_{22,1} - D_{m,1}, \qquad (30)$$

$$A_{m+1,2} := m(\beta_m^{-1})_{11,2} - A_{m,2}, \quad B_{m+1,2} := m(\beta_m^{-1})_{12,2} - B_{m,2}. \qquad (31)$$

Observing that

$$\det\beta_{m+1} = \begin{vmatrix} mS_D(\beta_m)_1^{-1} & B_{m+1,0}\\ -mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1} & D_{m+1,0}\end{vmatrix}\epsilon^{-r} + O(\epsilon^{-r+1}), \quad \epsilon\to 0,$$

the result follows. □

Now observe that

$$
\begin{aligned}
Z_1 := &\ mD_{m,0}^{-1} + mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} - D_{m-1} - D_{m,0} - \alpha_{22}\\
&- (-mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1})(mS_D(\beta_m)_1^{-1})^{-1}(-mS_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} - B_{m-1} - \alpha_{12})\\
= &\ mD_{m,0}^{-1} - D_{m-1} - D_{m,0} - \alpha_{22} - D_{m,0}^{-1}C_{m,0}(B_{m-1} + \alpha_{12}).
\end{aligned}
$$

Using the determinant expansion in Schur complements of lemma 2, one observes that

$$\begin{vmatrix} mS_D(\beta_m)_1^{-1} & -mS_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} - B_{m-1} - \alpha_{12} \\ -mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1} & mD_{m,0}^{-1} + mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} - D_{m-1} - D_{m,0} - \alpha_{22} \end{vmatrix}$$
$$= \det\left(mS_D(\beta_m)_1^{-1}\right)\det Z_1.$$

and the first point of the theorem is proved.

Let us now go one step further in the discrete matrix chain and move to position $m+2$.

Lemma 3. *Whenever* $\det D_{m,0} \neq 0$ *and* $\det Z_1 \neq 0$ *the following asymptotic hold.*

$$\det \beta_{m+2} = \epsilon^{-r}\begin{vmatrix} -mS_D(\beta_m)_1^{-1} & mS_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} + B_{m-1} \\ mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1} & (m+1)Z_1^{-1} - mD_{m,0}^{-1} \\ & \quad -mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} + D_{m-1} \end{vmatrix}$$
$$+ O(\epsilon^{-r+1})$$

for $\epsilon \to 0$.

Proof. As $\det \beta_{m+1} = O(\epsilon^{-r})$, $\epsilon \to 0$, and consequently point (2) of proposition 1 tells us that $\beta_{m+1}^{-1} \in \mathcal{A}_{\mathfrak{K}}$. Therefore, the following asymptotic expansion for the inverse matrix holds

$$\beta_{m+1}^{-1} = \begin{pmatrix} 0 & 0 \\ (\beta_{m+1}^{-1})_{21,0} & (\beta_{m+1}^{-1})_{22,0} \end{pmatrix} + \begin{pmatrix} (\beta_{m+1}^{-1})_{11,1} & (\beta_{m+1}^{-1})_{12,1} \\ (\beta_{m+1}^{-1})_{21,1} & (\beta_{m+1}^{-1})_{22,1} \end{pmatrix}\epsilon$$
$$+ \begin{pmatrix} (\beta_{m+1}^{-1})_{11,2} & (\beta_{m+1}^{-1})_{12,2} \\ (\beta_{m+1}^{-1})_{21,2} & (\beta_{m+1}^{-1})_{22,2} \end{pmatrix}\epsilon^2 + O(\epsilon^3), \tag{32}$$

for $\epsilon \to 0$. Here the blocks $(\beta_{m+1}^{-1})_{ab,j}$ are to be found from the asymptotic expansion (26). We conclude

$$(\beta_{m+1}^{-1})_{21,0} = Z_1^{-1}D_{m,0}^{-1}C_{m,0}, \quad (\beta_{m+1}^{-1})_{22,0} = Z_1^{-1},$$
$$(\beta_{m+1}^{-1})_{11,1} = \frac{1}{m}S_D(\beta_m)_1 - \frac{1}{m}S_D(\beta_m)_1B_{m+1,0}Z_1^{-1}D_{m,0}^{-1}C_{m,0},$$
$$(\beta_{m+1}^{-1})_{12,1} = -\frac{1}{m}S_D(\beta_m)_1B_{m+1,0}Z_1^{-1},$$
$$(\beta_{m+1}^{-1})_{21,1} = -Z_1^{-1}D_{m,0}^{-1}C_{m,0} - \frac{1}{m}Z_1^{-1}(C_{m+1,0} + D_{m,0}^{-1}C_{m,0}A_{m+1,0})S_D(\beta_m)_1$$
$$\times(\mathbb{I}_r - B_{m+1,0}Z_1^{-1}D_{m,0}^{-1}C_{m,0}),$$
$$(\beta_{m+1}^{-1})_{22,1} = -Z_1^{-1} + \frac{1}{m}Z_1^{-1}(C_{m+1,0} + D_{m,0}^{-1}C_{m,0}A_{m+1,0})S_D(\beta_m)_1B_{m+1,0}Z_1^{-1},$$
$$(\beta_{m+1}^{-1})_{11,2} = -\frac{1}{m^2}S_D(\beta_m)_1A_{m+1,0}S_D(\beta_m)_1 + \frac{1}{m^2}S_D(\beta_m)_1A_{m+1,0}S_D(\beta_m)_1B_{m+1,0}Z_1^{-1}D_{m,0}^{-1}C_{m,0}$$
$$+\frac{1}{m^2}S_D(\beta_m)_1B_{m+1,0}Z_1^{-1}(C_{m+1,0} + D_{m,0}^{-1}C_{m,0}A_{m+1,0})S_D(\beta_m)_1(\mathbb{I}_r - B_{m+1,0}Z_1^{-1}D_{m,0}^{-1}C_{m,0}),$$
$$(\beta_{m+1}^{-1})_{12,2} = -\frac{1}{m^2}S_D(\beta_m)_1B_{m+1,0}Z_1^{-1}(C_{m+1,0} + D_{m,0}^{-1}C_{m,0}A_{m+1,0})S_D(\beta_m)_1B_{m+1,0}Z_1^{-1}$$
$$+\frac{1}{m^2}S_D(\beta_m)_1A_{m+1,0}S_D(\beta_m)_1B_{m+1,0}Z_1^{-1}.$$

If we substitute equations (27)–(31) into equation (1), we have that for $\epsilon \to 0$

$$\beta_{m+2} = \begin{pmatrix} -mS_D(\beta_m)_1^{-1} & 0 \\ mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1} & 0 \end{pmatrix}\epsilon^{-1} + \begin{pmatrix} A_{m+2,0} & B_{m+2,0} \\ C_{m+2,0} & D_{m+2,0} \end{pmatrix} + \begin{pmatrix} A_{m+2,1} & B_{m+2,1} \\ C_{m+2,1} & D_{m+2,1} \end{pmatrix}\epsilon + O(\epsilon^2), \tag{33}$$

where

$$\begin{aligned}
A_{m+2,0} &:= -A_{m+1,0} - \alpha_{11}, \qquad B_{m+2,0} := -B_{m+1,0} - \alpha_{12},\\
C_{m+2,0} &:= (m+1)(\beta_{m+1}^{-1})_{21,0} - C_{m+1,0} - C_{m,0} - \alpha_{21},\\
&\qquad D_{m+2,0} := (m+1)(\beta_{m+1}^{-1})_{22,0} - D_{m+1,0} - D_{m,0} - \alpha_{22},\\
A_{m+2,1} &:= (m+1)(\beta_{m+1}^{-1})_{11,1} - A_{m+1,1} - A_{m,1},\\
&\qquad B_{m+2,1} := (m+1)(\beta_{m+1}^{-1})_{12,1} - B_{m+1,1} - B_{m,1},\\
C_{m+2,1} &:= (m+1)(\beta_{m+1}^{-1})_{21,1} - C_{m+1,1} - C_{m,1},\\
&\qquad D_{m+2,1} := (m+1)(\beta_{m+1}^{-1})_{22,1} - D_{m+1,1} - D_{m,1},\\
A_{m+2,2} &:= (m+1)(\beta_{m+1}^{-1})_{11,2} - A_{m+1,2} - A_{m,2},\\
&\qquad B_{m+2,2} := (m+1)(\beta_{m+1}^{-1})_{12,2} - B_{m+1,2} - B_{m,2}.
\end{aligned}$$

Now, observing that

$$\det \beta_{m+2} = \begin{vmatrix} -mS_D(\beta_m)_1^{-1} & B_{m+2,0} \\ mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1} & D_{m+2,0} \end{vmatrix}\epsilon^{-r} + O(\epsilon^{-r+1}), \qquad \epsilon \to 0,$$

the result follows. □

Note that

$$\begin{aligned}
Z_2 := &(m+1)(mD_{m,0}^{-1} - D_{m,0}^{-1}C_{m,0}(B_{m-1} + \alpha_{12}) - D_{m-1} - D_{m,0} - \alpha_{22})^{-1}\\
&+ D_{m,0}^{-1}C_{m,0}B_{m-1} - mD_{m,0}^{-1} + D_{m-1}.
\end{aligned}$$

We expand the determinant according to Schur complements, obtaining

$$\begin{aligned}
&\begin{vmatrix} -mS_D(\beta_m)_1^{-1} & mS_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} + B_{m-1} \\ mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1} & (m+1)Z_1^{-1} - mD_{m,0}^{-1} - mD_{m,0}^{-1}C_{m,0}S_D(\beta_m)_1^{-1}B_{m,1}D_{m,0}^{-1} + D_{m-1} \end{vmatrix}\\
&\qquad = \det\Big(- mS_D(\beta_m)_1^{-1}\Big)\det Z_2
\end{aligned}$$

from which the second point of the theorem follows immediately.

Lemma 4. *Assuming that* $\det D_{m,0} \neq 0$*,* $\det Z_1 \neq 0$ *and* $\det Z_2 \neq 0$ *the following asymptotic expansion for* $\epsilon \to 0$ *holds*

$$\begin{aligned}
\det \beta_{m+3} = \epsilon^r &\begin{vmatrix} (m+2)(\beta_{m+2}^{-1})_{11,1} & (m+2)(\beta_{m+2}^{-1})_{12,1} \\ \quad -(m+1)(\beta_{m+1}^{-1})_{11,1} + A_{m,1} & \quad -(m+1)(\beta_{m+1}^{-1})_{12,1} + B_{m,1} \\ (m+2)(\beta_{m+2}^{-1})_{21,0} & (m+2)(\beta_{m+2}^{-1})_{22,0} \\ \quad -(m+1)(\beta_{m+1}^{-1})_{21,0} + C_{m,0} & \quad -(m+1)(\beta_{m+1}^{-1})_{22,0} + D_{m,0} \end{vmatrix}\\
&+ O(\epsilon^{r+1}),
\end{aligned}$$

where

$$\begin{aligned}
&(\beta_{m+2}^{-1})_{21,0} := Z_2^{-1}D_{m,0}^{-1}C_{m,0}, \qquad (\beta_{m+2}^{-1})_{22,0} := Z_2^{-1},\\
&(\beta_{m+2}^{-1})_{11,1} := -\frac{1}{m}S_D(\beta_m)_1(\mathbb{I}_r - B_{m+2,0}Z_2^{-1}D_{m,0}^{-1}C_{m,0}), \quad (\beta_{m+2}^{-1})_{12,1} := \frac{1}{m}S_D(\beta_m)_1B_{m+2,0}Z_2^{-1},
\end{aligned}$$

$$
\begin{aligned}
(\beta_{m+2}^{-1})_{21,1} &:= -Z_2^{-1} D_{m,0}^{-1} C_{m,0} + \frac{1}{m} Z_2^{-1} (C_{m+2,0} + D_{m,0}^{-1} C_{m,0} A_{m+2,0}) S_D(\beta_m)_1 \\
&\quad \times (\mathbb{I}_r - B_{m+2,0} Z_2^{-1} D_{m,0}^{-1} C_{m,0}), \\
(\beta_{m+2}^{-1})_{22,1} &:= -Z_2^{-1} - \frac{1}{m} Z_2^{-1} (C_{m+2,0} + D_{m,0}^{-1} C_{m,0} A_{m+2,0}) S_D(\beta_m)_1 B_{m+2,0} Z_2^{-1}, \\
(\beta_{m+2}^{-1})_{11,2} &:= \frac{1}{m^2} S_D(\beta_m)_1 B_{m+2,0} Z_2^{-1} (C_{m+2,0} + D_{m,0}^{-1} C_{m,0} A_{m+2,0}) S_D(\beta_m)_1 \\
&\quad \times (\mathbb{I}_r - B_{m+2,0} Z_2^{-1} D_{m,0}^{-1} C_{m,0}) \\
&\quad - \frac{1}{m^2} S_D(\beta_m)_1 A_{m+2,0} S_D(\beta_m)_1 (\mathbb{I}_r - B_{m+2,0} Z_2^{-1} D_{m,0}^{-1} C_{m,0}), \\
(\beta_{m+2}^{-1})_{12,2} &:= \frac{1}{m^2} S_D(\beta_m)_1 A_{m+2,0} S_D(\beta_m)_1 B_{m+2,0} Z_2^{-1} \\
&\quad - \frac{1}{m^2} S_D(\beta_m)_1 B_{m+2,0} Z_2^{-1} (C_{m+2,0} + D_{m,0}^{-1} C_{m,0} A_{m+2,0}) S_D(\beta_m)_1 B_{m+2,0} Z_2^{-1}.
\end{aligned}
$$

Proof. From equation (33) we obtain that $\beta_{m+2} \in \mathbb{L}$. Therefore, since $\det Z_2 \neq 0$, we have

$$
\begin{aligned}
\beta_{m+2}^{-1} &= \begin{pmatrix} 0 & 0 \\ (\beta_{m+2}^{-1})_{21,0} & (\beta_{m+2}^{-1})_{22,0} \end{pmatrix} + \begin{pmatrix} (\beta_{m+2}^{-1})_{11,1} & (\beta_{m+2}^{-1})_{12,1} \\ (\beta_{m+2}^{-1})_{21,1} & (\beta_{m+2}^{-1})_{22,1} \end{pmatrix} \epsilon \\
&\quad + \begin{pmatrix} (\beta_{m+2}^{-1})_{11,2} & (\beta_{m+2}^{-1})_{12,2} \\ (\beta_{m+2}^{-1})_{21,2} & (\beta_{m+2}^{-1})_{22,2} \end{pmatrix} \epsilon^2 + O(\epsilon^3),
\end{aligned} \tag{34}
$$

where the blocks $(\beta_{m+2}^{-1})_{ab,j}$ are determined by the asymptotic expansion (33). If we substitute (26), (33) and (34) into the matrix equation (1), we have that

$$
\beta_{m+3} = \begin{pmatrix} 0 & 0 \\ C_{m+3,0} & D_{m+3,0} \end{pmatrix} + \begin{pmatrix} A_{m+3,1} & B_{m+3,1} \\ C_{m+3,1} & D_{m+3,1} \end{pmatrix} \epsilon + \begin{pmatrix} A_{m+3,2} & B_{m+3,2} \\ C_{m+3,2} & D_{m+3,2} \end{pmatrix} \epsilon^2 + O(\epsilon^3),
$$

where

$$
\begin{aligned}
C_{m+3,0} &:= (m+2)(\beta_{m+2}^{-1})_{21,0} - (m+1)(\beta_{m+1}^{-1})_{21,0} + C_{m,0}, \\
D_{m+3,0} &:= (m+2)(\beta_{m+2}^{-1})_{22,0} - (m+1)(\beta_{m+1}^{-1})_{22,0} + D_{m,0}, \\
A_{m+3,1} &:= (m+2)(\beta_{m+2}^{-1})_{11,1} - (m+1)(\beta_{m+1}^{-1})_{11,1} + A_{m,1}, \\
B_{m+3,1} &:= (m+2)(\beta_{m+2}^{-1})_{12,1} - (m+1)(\beta_{m+1}^{-1})_{12,1} + B_{m,1}, \\
C_{m+3,1} &:= (m+2)(\beta_{m+2}^{-1})_{21,1} - (m+1)(\beta_{m+1}^{-1})_{21,1} + C_{m,1}, \\
D_{m+3,1} &:= (m+2)(\beta_{m+2}^{-1})_{22,1} - (m+1)(\beta_{m+1}^{-1})_{22,1} + D_{m,1}, \\
A_{m+3,2} &:= (m+2)(\beta_{m+2}^{-1})_{11,2} - (m+1)(\beta_{m+1}^{-1})_{11,2} + A_{m,2}, \\
B_{m+3,2} &:= (m+2)(\beta_{m+2}^{-1})_{12,2} - (m+1)(\beta_{m+1}^{-1})_{12,2} + B_{m,2}.
\end{aligned}
$$

Then, if we use again proposition 1, we deduce

$$
\det \beta_{m+3} = \epsilon^r \begin{vmatrix} A_{m+3,1} & B_{m+3,1} \\ C_{m+3,0} & D_{m+3,0} \end{vmatrix} + O(\epsilon^{r+1}), \qquad \epsilon \to 0, \tag{35}
$$

and the result follows. □

Note that

$$
Z_3 = D_{m,0} - (m+1) Z_1^{-1} + (m+2) Z_2^{-1}.
$$

Note the similarity with equation (17).

Taking into account that

$$C_{m+3,0} = Z_3 D_{m,0}^{-1} C_{m,0}, \qquad D_{m+3,0} = Z_3, \tag{36}$$

we express the determinant in equation (35) as follows:

$$\begin{vmatrix} A_{m+3,1} & B_{m+3,1} \\ C_{m+3,0} & D_{m+3,0} \end{vmatrix} = \det Z_3 \det(A_{m+3,1} - B_{m+3,1} D_{m,0}^{-1} C_{m,0}), \tag{37}$$

where

$$A_{m+3,1} - B_{m+3,1} D_{m,0}^{-1} C_{m,0} = -\frac{(m+3)}{m} S_D(\beta_m)_1.$$

This implies that the determinant in equation (35) vanishes if and only if

$$\det Z_3 = 0.$$

Finally, under the previous hypotheses, equations (9)–(11) hold. As a by product of the proof of theorem 1, we obtain that

$$\beta_{m+4} = \beta_{m+3}^{-1} A - \beta_{m+3} - \alpha,$$

where $\beta_{m+3}, A \in \mathcal{A}_{\mathfrak{K}}$ and $(\beta_{m+3})^{-1} \in \epsilon^{-1}\mathcal{A}_{\mathfrak{L}}$. According to proposition 1 (6), $\beta_{m+3}^{-1} A \in \mathcal{A}$, so that we can write

$$\beta_{m+4} = O(1), \qquad \epsilon \to 0.$$

We can write the matrix dynamical system (1) as

$$\beta_{n-1} = n\beta_n^{-1} - \beta_{n+1} - \beta_n - \alpha, \tag{38}$$

which can be seen as the application of a *time reversal symmetry*. From $\beta_{m+4} \in \mathcal{A}$ and $\beta_{m+3} \in \mathcal{A}_{\mathfrak{K}}$, understood now as initial conditions, we obtain the quantities β_{m+2}, β_{m+1}, β_m and β_{m-1}. Observe that our initial assumption was precisely that $\beta_{m-1} \in \mathcal{A}$ and $\beta_m \in \mathcal{A}_{\mathfrak{K}}$, see (7). Hence, the whole forward process, and its conclusions about the asymptotic behaviours, can be reversed backwards. Consequently, since the assumption that $\det \beta_{m+4,0} = 0$ reduces the number of free parameters from N^2 to $N^2 - 1$, we conclude that β_{m-1} involves at most $N^2 - 1$ free parameters (if no further constraint is requested). This is in contradiction to our departing hypothesis that β_{m-1} has N^2 free parameters. Therefore $\det \beta_{m+4} = O(1)$ as $\epsilon \to 0$ generically.

Acknowledgments

PT has been supported by Spanish 'Ministerio de Ciencia e Innovación' grant FIS2011–00260. GC-C benefitted from the financial support of a 'Acción Especial' Ref. AE1/13-18837 of the Universidad Complutense de Madrid. MM acknowledges economical support from the Spanish 'Ministerio de Economía y Competitividad' research project MTM2012-36732-C03-01, *Ortogonalidad y aproximacion; Teoria y Aplicaciones*.

References

[1] Ablowitz M J, Halburd R and Herbst B 2000 On the extension of the Painlevé property to difference equations *Nonlinearity* **13** 889–905
[2] Adler M, van Moerbeke P and Vanhaecke P 2008 Singularity confinement for a class of m-th order difference equations of combinatorics *Phil. Trans. R. Soc. Lond.* A **366** 877-922
[3] Arinkin D and Borodin A 2006 Moduli spaces of d-connections, difference Painlevé equations *Duke Math. J.* **134** 515–56
[4] Bellon M P and Viallet C-M 1999 Algebraic entropy *Commun. Math. Phys.* **204** 425–37

[5] Bobenko A I and Suris Y 2008 Discrete differential geometry. Integrable structure *Graduate Studies in Mathematics* vol 98 (Providence, RI: American Mathematical Society) xxiv+404pp

[6] Bobenko A I and Suris Y 2002 Integrable systems on quad-graphs *Int. Math. Res. Not.* **11** 573–611

[7] Cassatella G A and Mañas M 2012 Riemann–Hilbert problems, matrix orthogonal polynomals, discrete matrix equations with singularity confinement *Stud. Appl. Math.* **128** 252–74

[8] Conte R (ed) 1999 *The Painlevé Property. One Century Later* (New York: Springer)

[9] Dynnikov I and Novikov S P 2003 Geometry of the triangle equation on two-manifolds *Moscow Math. J.* **3** 419–38

[10] Fokas A S, Its A R and Kitaev A V 1991 Discrete Painlevé equations, their appearance in quantum gravity *Commun. Math. Phys.* **142** 313–44

[11] Freud G 1976 On the coefficients in the recursion formulae of orthogonal polynomials *Proc. R. Irish Acad.* A **76** 1–6

[12] Gelfand I, Gelfand S, Retakh V and Wilson R 2005 Quasideterminants *Adv. Math.* **193** 56–141

[13] Grammaticos B, Ramani A and Papageorgiou V 1991 Do integrable mappings have the Painlevé property? *Phys. Rev. Lett.* **67** 1825–8

[14] Grünbaum F A, de la Iglesia M D and Martinez-Finkelshtein A 2011 Properties of matrix orthogonal polynomials via their Riemann–Hilbert characterization *SIGMA* **7** 098

[15] Hietarinta J and Viallet C 1998 Singularity confinement, chaos in discrete systems *Phys. Rev. Lett.* **81** 325–8

[16] 't Hooft G 1996 Quantization of point particles in (2+1)-dimensional gravity and spacetime discreteness *Class. Quantum Grav.* **13** 1023–39

[17] Lafortune S, Ramani A, Grammaticos B, Ohta Y and Tamizhmani K M 2001 Blending two discrete integrability criteria: singularity confinement and algebraic entropy *Bäcklund and Darboux transformations. The geometry of solitons (Halifax, NS, 1999)* (*CRM Proceedings and Lecture Notes* vol 29 (Providence, RI: American Mathematical Society) pp 299–311

[18] Moser J and Veselov A P 1991 Discrete versions of some classical integrable systems, factorization of matrix polynomials *Commun. Math. Phys.* **139** 217–43

[19] Newman M E J 2010 *Networks* (Oxford: Oxford University Press)

[20] Novikov S P and Shvarts A S 1999 Discrete Lagrangian systems on graphs. Symplecto-topological properties *Usp. Mat. Nauk* **54** 257–58 (in Russian)
Novikov S P and Shvarts A S 1999 *Math. Surv.* **54** 258–59 (Engl. transl.)

[21] Painlevé P 1973 *Leçons sur la théorie analytique des équations différentielles (Leçons de Stockholm, delivered in 1895)* Hermann, Paris (1897). Reprinted in Œuvres de Paul Painlevé, vol I, Éditions du CNRS, Paris

[22] Ramani A, Grammaticos B, Tamizhmani T and Tamizhmani K M 2003 The road to the discrete analogue of the Painlevé property: Nevanlinna meets singularity confinement *Comput. Math. Appl.* **45** 1001–12

[23] Suris Yu B 2003 *The Problem of Integrable Discretization: Hamiltonian Approach* (*Progress in Mathematics* vol 219) (Basel: Birkhäuser)

[24] Tempesta P 2013 Integrable maps from Galois differential algebras, Borel transforms, number sequences *J. Diff. Eqns* **255** 2981–95

[25] Tsuda T 2009 Universal character and q-difference Painlevé equations *Math. Ann.* **345** 395–415

CHAPTER IV

FREUD POLYNOMIALS ON THE CIRCLE
AND
A MATRIX PAINLEVÉ II
DISCRETE EQUATION

FREUD POLYNOMIALS ON THE CIRCLE AND A MATRIX PAINLEVÉ II DISCRETE EQUATION

GIOVANNI A. CASSATELLA-CONTRA, MANUEL MAÑAS BAENA, AND PIERGIULIO TEMPESTA

ABSTRACT. The matrix Riemann-Hilbert problem associated with matrix orthogonal polynomials on the unit cirle is formulated. For a class of matrix Freud polynomials, the recursion coefficients are studied. This approach allows to derive in a simple way a matrix discrete version of the Painlevé equation II. The singularity properties of this matrix models are analyzed in some particular cases, where it is shown that the singularity confinement property holds.

CONTENTS

Date: October 9, 2015.

1. Introduction

The purpose of this paper is to explore the connection between the theory of integrable discrete equations of Painlevé type and the Riemann-Hilbert problem for matrix orthogonal polynomials.

In the seminal paper [9], it was observed that a matrix 2×2 Riemann-Hilbert problem led naturally to a class of orthogonal polynomials on the real line.

In the present work, we shall extend the Riemann-Hilbert approach to the unit circle, for a class of orthogonal polynomials of Szegő type defined in terms of a matrix measure.

The recursion relations associated with the problem leads naturally to an integrable discrete equation, which is the discrete version of the Painlevé equation. This equation was obtained by van Assche by means of a different analytic approach.

An interesting problem is to relate the integrability properties of difference equations with their singularity properties. In [10], the singularity confinement was proposed as a discrete version of the Painlevé property. The confinement property amounts to require that if a singularity appears at some value of the lattice where the equation is defined, then it would disappear after performing a sufficient (finite) number of iterations.

The notion of algebraic entropy [5], [12] or an approach based on Nevalinna theory [1], [17], have also been proposed as alternatives to the singularity analysis.

Although, as observed in [2], a large family of difference equations coming from unitary integrals and combinatorics possess the confinement property, it is also clear that it is not a sufficient condition for integrability for all classes of equations [11].

In [6], [7], the singularity analysis for a matrix discrete version of the Painlevé I equation was performed. It was found that the singularity confinement holds generically, i.e. in the whole space of parameters except possibly for algebraic subvarieties.

Therefore, it is very natural to further explore to which extent this property is intimately related to discrete integrability in the case of *matrix integrable models.*

To this aim, we shall also perform an analysis of singularity confinement for the matrix discrete Painlevé equation arising from the Riemann-Hilbert problem. We shall prove that for the 2×2 case, there is indeed confinement under generic conditions.

An open problem is the generalization of the approach proposed in this work to the case of non-Hermitian matrix measures. We surmise that this case would lead to interesting new classes of matrix integrable models.

2. The Riemann–Hilbert approach

In this section we will study a matrix version of the Riemann–Hilbert problem, defined on the unit circle.

2.1. **Matrix Szegő polynomials on the unit circle.** We will denote the unit circle by $\mathbb{T} := \{z \in \mathbb{C} :| z |= 1\}$. We shall consider a matrix measure μ supported in $\mathbb{T}$, that satisfies $d\mu = w(z)\frac{dz}{iz}$. Here $w(z)$ is a continuous and hermitian $N \times N$ matrix that is defined in $\mathbb{T}$, and can be expanded analytically in an annulus around $\mathbb{T}$.

Given the weight w we will suppose that the following left and right *monic matrix Szegő polynomials* P_n^L and P_n^R exist and satisfy the following orthogonality relations:

$$\int_{\mathbb{T}} P_n^L(z) z^{-j} d\mu = -i \int_{\mathbb{T}} P_n^L(z) z^{-j-1} w(z) dz = 0, \qquad j = 0, \dots, n-1, \tag{1}$$

and

$$\int_{\mathbb{T}} d\mu P_n^R(z) z^{-j} = -i \int_{\mathbb{T}} w(z) P_n^R(z) z^{-j-1} dz = 0, \qquad j = 0, \dots, n-1, \tag{2}$$

respectively.
We can also define the *reverse Szegő polynomials* as

$$\tilde{P}_n^{L,R}(z) := z^n [P_n^{L,R}(1/\overline{z})]^*.$$

The reverse left Szegő polynomials satisfy the following orthogonality relations

$$\int_{\mathbb{T}} \tilde{P}_n^L(z) z^{-j} d\mu = -i \int_{\mathbb{T}} \tilde{P}_n^L(z) z^{-j-1} w(z) dz = 0, \qquad j = 1, \dots, n, \tag{3}$$

and similarly for the reverse right polynomials. We shall also introduce the notations

$$Q_n^L(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{P_n^L(u) w(u)}{u^n (u-z)} du$$

for the Cauchy transforms of $P_n^L(z)$, and

$$\tilde{Q^L}_n(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{w(u) \tilde{P^L}_n(u)}{u^{n+1} (u-z)} du$$

the reverse Cauchy transforms of P_n^L. Analogous formulae hold for Q^R and $\tilde{Q^R}$ in terms of P_n^R and $\tilde{P_n^R}$, respectively. Notice that

$$\tilde{Q}_n^{L,R}(z) := -z^{-n-1} [Q_n^{L,R}(1/\overline{z})]^*. \tag{4}$$

2.2. **The Riemann-Hilbert problem.**

Definition 1. *We introduce the Riemann-Hilbert problem consisting in the determination of a $2N \times 2N$ matrix function $Y_n(z) \in \mathbb{C}^{2N \times 2N}$ such that*

1. *$Y_n(z)$ is analytic in $\mathbb{C} \setminus \mathbb{T}$.*
2. *For $z \in \mathbb{T}$, Y_n satisfies the jumping condition*

$$Y_{n_+}(z) = Y_{n_-}(z) \begin{pmatrix} \mathbb{I}_N & w(z) z^{-n} \\ 0 & \mathbb{I}_N \end{pmatrix}.$$

(3) *Asymptotically, it behaves as*

$$Y_n(z) = (\mathbb{I}_{2N} + O(z^{-1}))\begin{pmatrix} \mathbb{I}_N z^n & 0 \\ 0 & \mathbb{I}_N z^{-n} \end{pmatrix} \qquad \text{for } z \to \infty.$$

We have the following result.

Theorem 1. *The unique solution of the Riemann-Hilbert problem stated in Definition 1 is represented by the matrix function* $Y_n(z)$

$$Y_n(z) := \begin{pmatrix} P^L_n(z) & Q^L_n(z) \\ \gamma_{n-1}\tilde{P^R}_{n-1}(z) & \gamma_{n-1}\tilde{Q^R}_{n-1}(z) \end{pmatrix}, \qquad n \geq 1, \tag{5}$$

$$Y_0(z) := \begin{pmatrix} \mathbb{I}_N & Q^L_0(z) \\ 0 & \mathbb{I}_N \end{pmatrix}, \qquad n = 0, \tag{6}$$

where the coefficients γ_n *are defined to be*

$$\gamma_n := -2\pi\Big(\int_{\mathbb{T}} \tilde{P^R}_n(z) d\mu\Big)^{-1} = -2\pi i\Big(\int_{\mathbb{T}} \tilde{P^R}_n(z) z^{-1} w(z) dz\Big)^{-1}. \tag{7}$$

Observe that the following relations hold

$$\gamma_n = -2\pi\Big(\Big[\int_{\mathbb{T}} d\mu P^L_n(z) z^{-n}\Big]^*\Big)^{-1} = -2\pi i\Big(\Big[\int_{\mathbb{T}} w(z) P^L_n(z) z^{-n-1} dz\Big]^*\Big)^{-1}. \tag{8}$$

2.3. **Recursion relations.** We wish to study here the properties of the recursion coefficients related with the Riemann-Hilbert problem defined above.

Definition 2. *The matrix* $Z_n(z)$ *is defined to be*

$$Z_n(z) := Y_n \begin{pmatrix} w(z) z^{-n} & 0 \\ 0 & \mathbb{I}_N \end{pmatrix}.$$

For $z \in \mathbb{T}$, Z_n, verifies the following jumping condition

$$Z_{n_+}(z) = Z_{n_-}(z) \begin{pmatrix} \mathbb{I}_N & \mathbb{I}_N \\ 0 & \mathbb{I}_N \end{pmatrix}. \tag{9}$$

It is remarkable that the jumping condition now is expressed in terms of a constant matrix. The matrix $Z_n(z)$ can also be regarded as the solution of a Riemann–Hilbert problem. Precisely, the following simple result holds.

Proposition 1. *The matrix function* $Z_n(z)$ *satisfies the following properties.*

(1) $Z_n(z)$ *is analytic in* $\mathbb{C} \setminus \{\mathbb{T} \cup 0\}$.
(2) *For* $z \in \mathbb{T}$, Z_n *satisfies the jumping condition* $Z_{n_+}(z) = Z_{n_-}(z) \begin{pmatrix} \mathbb{I}_N & \mathbb{I}_N \\ 0 & \mathbb{I}_N \end{pmatrix}$.
(3) $Z_n(z) = (\mathbb{I}_{2N} + O(z^{-1}))\begin{pmatrix} w(z) & 0 \\ 0 & z^{-n}\mathbb{I}_N \end{pmatrix}$ *for* $z \to \infty$.

From the properties of $Y_n(z)$ we can deduce that

i) $\det Y_n(z)$ is analytic in $\mathbb{C} \setminus \mathbb{T}$.

ii) $\det Y_{n_+}(z) = \det Y_{n_-}(z)$.

iii) $\det Y_n(z) \to (1 + O(z^{-1}))$ for $z \to \infty$.

Hence, we obtain that $\det Y_n(z) = 1$ and $\det Z_n(z) = z^{-Nn} \det w(z)$. From this we also deduce that $Y_n(z)^{-1}$ exists and is analytic in $\mathbb{C} \setminus \mathbb{T}$. If the weight $w(z)$ does not have any zeros in $\mathbb{C}$ then we conclude that $Z_n(z)^{-1}$ exists and is analytic in $\mathbb{C} \setminus \{\mathbb{T} \cup 0\}$.

Definition 3. *We introduce the auxiliary matrices*

$$R_n(z) := Z_{n+1}(z) Z_n(z)^{-1},$$

$$M_n(z) := \frac{dZ_n(z)}{dz} Z_n(z)^{-1}.$$

Notice that $R_n(z)$ solves the following Riemann-Hilbert problem

(1) $R_n(z)$ is analytic in $\mathbb{C} \setminus \{\mathbb{T} \cup 0\}$.

(2) For $z \in \mathbb{T}$ $R_{n+}(z) = R_{n-}(z)$.

(3) $R_n(z) = \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & 0 \end{pmatrix} + O(z^{-1})$ for $z \to \infty$.

Definition 4. *Given $Y_n(z)$, we define the matrix $S_n(z)$ from the relation*

$$Y_n(z) = S_n(z) \begin{pmatrix} z^n \mathbb{I}_N & 0 \\ 0 & z^{-n} \mathbb{I}_N \end{pmatrix}.$$

The asymptotic behaviour of $Y_n(z)$ implies that

$$S_n(z) = \mathbb{I}_{2N} + \beta_n^{(1)} z^{-1} + \beta_n^{(2)} z^{-2} + O(z^{-3}), \qquad z \to \infty,$$

where $\beta_n^{(i)}, i = 0, 1, 2, \ldots$, are $2N \times 2N$ matrices. By representing them in the form

$$\beta_n^{(i)} := \begin{pmatrix} a_{n,i} & b_{n,i} \\ c_{n,i} & d_{n,i} \end{pmatrix}, \ i \geq 0,$$

we obtain for $Y_n(z)$ the asymptotic expression

$$Y_n(z) = \begin{pmatrix} z^n \mathbb{I}_N + a_{n,1} z^{n-1} + O(z^{n-2}) & b_{n,1} z^{-n-1} + b_{n,2} z^{-n-2} + O(z^{-n-3}) \\ c_{n,1} z^{n-1} + c_{n,2} z^{n-2} + O(z^{n-3}) & z^{-n} \mathbb{I}_N + d_{n,1} z^{-n-1} + O(z^{-n-2}) \end{pmatrix}, \tag{10}$$

From (5), (6) and (10) we have that

$$a_{n,i} = c_{n,i} = 0, \ i > n, \tag{11}$$

$$d_{0,i} = 0, \ i \geq 1. \tag{12}$$

From the definition of $Q_n^L(z)$ we have that

$$Q_n^L(z) = b_{n,1} z^{-n-1} + O(z^{-n-2}), \qquad z \to \infty,$$

where

$$b_{n,1} = \frac{-1}{2\pi i} \int_{\mathbb{T}} dz w(z) P_n^L(z). \tag{13}$$

2.3.1. *Recursion relations for the coefficients of $R_n(z)$.* . By combining the previous definitions and taking into account the fact that $Y_n(z)$ and $Y_{n+1}(z)$ are analytic, we get that $R_n(z)$ can only have at most a simple pole at $z = 0$, that is

$$R_n(z) = Y_{n+1}(z) \begin{pmatrix} \mathbb{I}_N z^{-1} & 0 \\ 0 & \mathbb{I}_N \end{pmatrix} Y_n(z)^{-1}\Big|_{\geq -1}.$$

Consequently, we have

$$\begin{aligned}
R_n(z) &:= S_{n+1}(z) \begin{pmatrix} z^{n+1}\mathbb{I}_N & 0 \\ 0 & \mathbb{I}_N z^{-n-1} \end{pmatrix} \begin{pmatrix} \mathbb{I}_N z^{-1} & 0 \\ 0 & \mathbb{I}_N \end{pmatrix} \begin{pmatrix} \mathbb{I}_N z^{-n} & 0 \\ 0 & z^n \mathbb{I}_N \end{pmatrix} S_n(z)^{-1} \\
&= (\mathbb{I}_{2N} + \beta_{n+1}^{(1)} z^{-1} + O(z^{-2})) \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & \mathbb{I}_N z^{-1} \end{pmatrix} (\mathbb{I}_{2N} - \beta_n^{(1)} z^{-1} + O(z^{-2})) \\
&= (\mathbb{I}_{2N} + \beta_{n+1}^{(1)} z^{-1} + O(z^{-2})) \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & \mathbb{I}_N z^{-1} \end{pmatrix} (\mathbb{I}_{2N} - \beta_n^{(1)} z^{-1} + O(z^{-2}))\Big|_{\geq -1} \\
&= \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & \mathbb{I}_N z^{-1} \end{pmatrix} - \begin{pmatrix} \mathbb{I}_N z^{-1} & 0 \\ 0 & 0 \end{pmatrix} \beta_n^{(1)} + \beta_{n+1}^{(1)} \begin{pmatrix} \mathbb{I}_N z^{-1} & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{I}_N + (a_{n+1,1} - a_{n,1}) z^{-1} & -z^{-1} b_{n,1} \\ c_{n+1,1} z^{-1} & z^{-1} \mathbb{I}_N \end{pmatrix}.
\end{aligned}$$

Notice that the matrix $R_n(z)$ can also be written as

$$R_n(z) = R_{n,0} + R_{n,1} z^{-1}, \tag{14}$$

where

$$R_{n,0} := \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & 0 \end{pmatrix}, \; R_{n,1} := \begin{pmatrix} a_{n+1,1} - a_{n,1} & -b_{n,1} \\ c_{n+1,1} & \mathbb{I}_N \end{pmatrix}.$$

In order to obtain recursion formulae from the coefficients of the matrix $R_n(z)$, observe that

$$Y_{n+1}(z) = R_n(z) Y_n(z) \begin{pmatrix} z\mathbb{I}_N & 0 \\ 0 & \mathbb{I}_N \end{pmatrix}, \tag{15}$$

so from the definition of $S_n(z)$ we have

$$S_{n+1}(z) = R_n(z) S_n(z) \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & z\mathbb{I}_N \end{pmatrix}. \tag{16}$$

2.3.2. *RH problem for $M_n(s)$.* . Now we are going to study the matrix $M_n(z)$. From its definition we have that

$$M_n(z) = \frac{dY_n(z)}{dz} Y_n(z)^{-1} + Y_n(z) \begin{pmatrix} (\frac{d}{dz}w)w^{-1} - nz^{-1}\mathbb{I}_N & 0 \\ 0 & 0 \end{pmatrix} Y_n(z)^{-1}. \tag{17}$$

Using the definition of $S_n(z)$ we have that

$$M_n(z) = \frac{dS_n(z)}{dz} S_n(z)^{-1} + S_n(z) K_n(z) S_n(z)^{-1}, \tag{18}$$

where

$$K_n(z) := \begin{pmatrix} (\frac{d}{dz}w)w^{-1} & 0 \\ 0 & -n\mathbb{I}_N z^{-1} \end{pmatrix}. \tag{19}$$

$M_n(z)$ satisfies the following conditions:

(1) $M_n(z)$ is analytic in $\mathbb{C} \setminus \{\mathbb{T} \cup 0\}$.
(2) For $z \in \mathbb{T}$, $M_{n+}(z) = M_{n-}(z)$.
(3) $M_n(z)$ has at least a simple pole at z=0.

3. Freud matrix polynomials and a discrete matrix Painlevé equation of type II

To relate the Riemann-Hilbert problem for Szegő polynomials with the theory of discrete matrix integrable systems, we shall specialize now our analysis to a class of Freud polynomials.

Precisely, we will choose as a weight $w(z) = e^{V(z)}$, where

$$V(z) := k(z + z^{-1}). \tag{20}$$

To ensure that $V(z)$ be Hermitian on the circle, k is also an Hermitian matrix.

From the previous discussion, we have deduced that $M_n(z)$ is analytic in $z \in \mathbb{C} \setminus \{0\}$, and from equation (17), that $M_n(z)$ has a pole of order 2.

Consequently, eq. (18) can be written as

(21)
$$M_n(z) = \Big[\frac{dS_n(z)}{dz} S_n(z)^{-1}\Big]_{\geq -2} + \Big[S_n(z) \begin{pmatrix} k - kz^{-2} & 0 \\ 0 & -nz^{-1}\mathbb{I}_N \end{pmatrix} S_n(z)^{-1}\Big]_{\geq -2},$$

where $\Big[\Big]_{\geq -2}$ means that we are only considering terms of order z^{-2} or higher. Substituting $S_n(z)$ in this equation we have that

$$M_n(z) = M_{n,2}z^{-2} + M_{n,1}z^{-1} + M_{n,0}, \tag{22}$$

where

$$M_{n,0} := \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix},$$

$$M_{n,1} := \begin{pmatrix} a_{n,1}k - ka_{n,1} & -kb_{n,1} \\ c_{n,1}k & -n\mathbb{I}_N \end{pmatrix},$$

$$M_{n,2} := \begin{pmatrix} M_{n,2,11} & M_{n,2,12} \\ (n-1)c_{n,1} - c_{n,1}ka_{n,1} + c_{n,2}k & -c_{n,1}kb_{n,1} - d_{n,1} \end{pmatrix},$$

where

$$M_{n,2,11} := kb_{n,1}c_{n,1} - k - a_{n,1} - ka_{n,2} + ka_{n,1}^2 - a_{n,1}ka_{n,1} + a_{n,2}k,$$
$$M_{n,2,12} := -(n+1)b_{n,1} + kb_{n,1}d_{n,1} - kb_{n,2} + ka_{n,1}b_{n,1} - a_{n,1}kb_{n,1}.$$

The aim of the subsequent analysis is to determine the matrix coefficients appearing in the asymptotic expression of $M_n(z)$.

3.1. Coefficients of $M_n(z)$.

3.1.1. *Coefficients* $a_{n,1}$, $b_{n,1}$ *and* $c_{n,1}$. From eq. (5), (6) and (15) we derive the following recurrence relations

$$\gamma_n \tilde{P^R}_n(z) = c_{n+1,1}P_n^L(z) + \gamma_{n-1}\tilde{P^R}_{n-1}(z), \tag{23}$$

$$P_{n+1}^L(z) = (z\mathbb{I}_N + a_{n+1,1} - a_{n,1})P_n^L(z) - b_{n,1}\gamma_{n-1}\tilde{P^R}_{n-1}(z), \tag{24}$$

for $n > 0$, and

$$P_1(z) = z\mathbb{I}_N + a_{1,1} - a_{0,1} = z\mathbb{I}_N + a_{1,1}, \tag{25}$$
$$c_{1,1} = \gamma_0, \tag{26}$$

for $n = 0$ (here we have substituted (11) in the second member of (25)). Now we substitute (23) in (24), and we get

$$P_{n+1}^L(z) = (z\mathbb{I}_N + a_{n+1,1} - a_{n,1} + b_{n,1}c_{n+1,1})P_n^L(z) - b_{n,1}\gamma_n\tilde{P^R}_n(z). \tag{27}$$

If we substitute (25) and (27) in (1) we have that

$$a_{1,1} = -b_{0,1}\gamma_0, \tag{28}$$

and

$$a_{n+1,1} - a_{n,1} = -b_{n,1}c_{n+1,1}, \tag{29}$$

respectively[1]. Then (25) and (27) can be written as

$$P^L_{n+1}(z) = zP^L_n(z) - \alpha_n \tilde{P^R}_n(z), \; n \geq 0, \tag{30}$$

where

$$\alpha_n := b_{n,1}\gamma_n. \tag{31}$$

In the Appendix, we shall prove that both α_n and γ_n are hermitian. From the relation $V(z) = V(z^{-1})$, it can be shown that also $b_{n,1}$ is hermitian, so $[b_{n,1}, \gamma_n] = 0$.

Another relevant aspect is that $P_n(z) := P^L_n(z) = P^R_n(z)$ (see equation (100)), and from (28) and (31) that

$$a_{1,1} = -\alpha_0. \tag{32}$$

Consequently,

$$P_{n+1}(z) = zP_n(z) - \alpha_n \tilde{P}_n(z), \; n \geq 0. \tag{33}$$

Then it is proven in the Appendix that for a $V(z)$ given in equation (20) we have that

$$[\alpha_n, \alpha_m] = 0, \tag{34}$$

$$[k, \alpha_n] = [k, \gamma_n] = 0, \tag{35}$$

$$[\gamma_n, \gamma_m] = [\alpha_n, \gamma_m] = 0, \; n, m \geq 0. \tag{36}$$

It is easy to see that (33) and (23) are equivalent if

$$c_{n,1} = -\gamma_{n-1}\alpha_{n-2}, \; n \geq 2, \tag{37}$$

and

$$\gamma^{-1}_{n+1}\gamma_n = \gamma_n\gamma^{-1}_{n+1} = \mathbb{I}_N - \alpha^2_n, \tag{38}$$

where we used equation (36) to pass from the first member to the second one of the last equation. If we substitute the definition of α_n and equation (37) in (29), we have that

[1] From (11) and (28) we can see that (29) is valid also for $n = 0$.

$$a_{n+1,1} - a_{n,1} = \alpha_n \alpha_{n-1}. \tag{39}$$

Note that this is valid only for $n \geq 1$. Summing up in n in this equation, we have that

$$a_{n,1} = \sum_{i=0}^{n-2} \alpha_{i+1}\alpha_i - \alpha_0,\ n \geq 2, \tag{40}$$

where we have used equation (32).

3.2. **Coefficients $d_{n,1}$, $b_{n,2}$ and $c_{n,2}$.** In this section we will obtain the expressions of the remaining coefficients of $S_n(z)$ that appear in $R_n(z)$ and $M_n(z)$, i.e. $d_{n,1}$, $b_{n,2}$ and $c_{n,2}$[2]. From equation (16) we have that

$$d_{n+1,1} - d_{n,1} = b_{n,1} c_{n+1,1}, \tag{41}$$

$$c_{n+1,2} = c_{n,1} + c_{n+1,1} a_{n,1}, \tag{42}$$

and

$$b_{n+1,1} = b_{n,2} + (a_{n+1,1} - a_{n,1}) b_{n,1} - b_{n,1} d_{n,1}. \tag{43}$$

Note that $d_{n+1,1} - d_{n,1} = -(a_{n+1,1} - a_{n,1})$, and if we substitute $n = 0$ in (41) and use (12), we have that

$$d_{1,1} = \alpha_0, \tag{44}$$

so

$$d_{n,1} = -\sum_{i=0}^{n-2} \alpha_{i+1}\alpha_i + \alpha_0,\ n \geq 2. \tag{45}$$

Since $d_{n,1} = -a_{n,1}$[3], and that

$$[b_{n,1}, d_{n,1}] = 0, \tag{46}$$

Equation (43) can be written as

$$b_{n,2} = b_{n+1,1} + b_{n,1} d_{n+1,1}. \tag{47}$$

[2] $a_{n,1}$, $b_{n,1}$, $c_{n,1}$ also appear in $R_n(z)$ and $M_n(z)$, but they have been obtained in last section

[3] It follows from equations (29), (41), (11) and (12).

If we substitute the expressions for $a_{n,1}$, $b_{n,1}$, $c_{n,1}$, $d_{n,1}$ and their initial conditions in equations (47) and (42), we have that

$$b_{n,2} = \alpha_{n+1}\gamma_{n+1}^{-1} - \alpha_n\gamma_n^{-1}\sum_{i=0}^{n-1}\alpha_{i+1}\alpha_i + \alpha_0\alpha_n\gamma_n^{-1},\ n \geq 1, \tag{48}$$

$$b_{0,2} = \alpha_1\gamma_1^{-1} + \alpha_0^2\gamma_0^{-1}, \tag{49}$$

and

$$c_{n,2} = -\alpha_{n-3}\gamma_{n-2} + \alpha_0\alpha_{n-2}\gamma_{n-1} - \alpha_{n-2}\gamma_{n-1}\sum_{i=0}^{n-3}\alpha_{i+1}\alpha_i,\ n \geq 3, \tag{50}$$

$$c_{2,2} = \gamma_0 + \alpha_0^2\gamma_1, \tag{51}$$

respectively. From equations (35), (34), (36), (46), and from the expressions of $a_{n,1}$, $b_{n,1}$, $c_{n,1}$, $d_{n,1}$, $b_{n,2}$ and $c_{n,2}$ written in terms of α_n and γ_n, we have that

$$[x, y] = 0, \tag{52}$$

$$[k, x] = 0, \tag{53}$$

where $x, y \in X$, with

$$X := \{\{a_{i,1}\}_{i=0}^{\infty}, \{b_{i,1}\}_{i=0}^{\infty}, \{c_{i,1}\}_{i=0}^{\infty}, \{d_{i,1}\}_{i=0}^{\infty}, \{b_{i,2}\}_{i=0}^{\infty}, \{c_{i,2}\}_{i=0}^{\infty}\}.$$

4. Recurrence matrix equation

It can be shown that

$$T\frac{d}{dz}Z_n(z) = \frac{d}{dz}TZ_n(z),$$

where T is such that $TF_n := F_{n+1}$. Since $R_n(z) = (TZ_n(z))Z_n(z)^{-1} = Z_{n+1}(z)Z_n(z)^{-1}$ and $M_n(z) = \frac{dZ_n(z)}{dz}Z_n(z)^{-1}$, then $R_n(z)$ and $M_n(z)$ must satisfy the following equation:

$$M_{n+1}(z)R_n(z) = \frac{dR_n(z)}{dz} + R_n(z)M_n(z). \tag{54}$$

If we substitute in equation (54) expressions (14) and (22), and we take into account equations (41), (42), (47), (52) and (53) we have that it reduces to

$$M_{n+1,2}R_{n,1} = R_{n,1}M_{n,2}, \tag{55}$$

i.e.

$$\begin{aligned}(kb_{n+1,1}c_{n+1,1} - a_{n+1,1})(a_{n+1,1} - a_{n,1}) + c_{n+1,1}(kb_{n+1,1}d_{n+1,1} - kb_{n+1,2} - (n+2)b_{n+1,1}) = \\ (kb_{n,1}c_{n,1} - a_{n,1})(a_{n+1,1} - a_{n,1}) - (n-1)b_{n,1}c_{n,1} + ka_{n,1}b_{n,1}c_{n,1} - kc_{n,2}b_{n,1},\end{aligned} \tag{56}$$

(57)
$$(-kb_{n+1,1}c_{n+1,1}+k+a_{n+1,1})b_{n,1}+kb_{n+1,1}d_{n+1,1}-kb_{n+1,2}-(n+2)b_{n+1,1}=\\ (kb_{n,1}d_{n,1}-kb_{n,2}-(n+1)b_{n,1})(a_{n+1,1}-a_{n,1})+kb_{n,1}^2c_{n,1}+b_{n,1}d_{n,1},$$

(58)
$$(a_{n+1,1}-a_{n,1})(nc_{n+1,1}-ka_{n+1,1}c_{n+1,1}+kc_{n+1,2})-kb_{n+1,1}c_{n+1,1}^2-c_{n+1,1}d_{n+1,1}=\\ c_{n+1,1}(kb_{n,1}c_{n,1}-k-a_{n,1})+(n-1)c_{n,1}-ka_{n,1}c_{n,1}+kc_{n,2},$$

and

$$b_{n,1}c_{n+1,1}+ka_{n+1,1}c_{n+1,1}b_{n,1}-kb_{n,1}c_{n+1,2}-kb_{n+1,1}c_{n+1,1}-d_{n+1,1}=\\ kc_{n+1,1}(b_{n,1}d_{n,1}-b_{n,2})-kb_{n,1}c_{n,1}-d_{n,1}. \tag{59}$$

If we substitute eqs. (32), (40), (44), (45) and (48)-(51) into eqs. (56)-(59), and we take into account eqs. (11) and (12), we obtain a set of equations, with their respective initial conditions, reported in Appendix 6.4. It can be proven that this set of equations holds valid if and only if α_n satisfies the recurrence

$$\alpha_{n+1}=-(n+1)k^{-1}(\mathbb{I}_N-\alpha_n^2)^{-1}\alpha_n-\alpha_{n-1},$$

(60)

with $\alpha_{-1}=-\mathbb{I}_N$ and $\alpha_0=b_{0,1}\gamma_0$ as initial conditions. This equation can be regarded as a matrix form of the discrete version of the second Painlevé equation.

5. Singularity confinement of a dPII

In this section we will study the singularity confinement of the matrix model (**??**). Notice that this model, by taking Hermitian conditions, a priori can be diagonalized, and it reduces to N copies of the same scalar equation.

In this section, we consider the matrix equation (**??**) *tout court*, i.e. as an abstract recurrence relation. We take as initial conditions

$$\alpha_{m-1}=\alpha_{m-1,0}+\alpha_{m-1,1}\epsilon+O(\epsilon^2), \tag{61}$$
$$\alpha_m=\alpha_{m,0}+\alpha_{m,1}\epsilon+O(\epsilon^2), \tag{62}$$

for $\epsilon\to 0$, where $\alpha_{m,0}$ is such that

$$\det(\mathbb{I}_N-\alpha_m^2)=\det(\mathbb{I}_N-\alpha_{m,0}^2+O(\epsilon))=O(\epsilon^s),\ s\in\mathbb{Z}. \tag{63}$$

5.1. **Scalar case (N=1).** We consider first the scalar model

$$\alpha_{n+1} = -(n+1)k^{-1}(1-\alpha_n^2)^{-1}\alpha_n - \alpha_{n-1}. \tag{64}$$

Conditions (61) and (62) can be written as

$$\alpha_{m-1} = A_0 + A_1\epsilon + O(\epsilon^2), \tag{65}$$

$$\alpha_m = a_0 + a_1\epsilon + O(\epsilon^2). \tag{66}$$

Note that if $a_0 = \pm 1$ we will have a singularity in α_{m+1}, so we assume that $a_0 = 1$ (with $a_0 = -1$ we would obtain a very similar result).
If we substitute (65) and (66) in (64), we have that

$$\alpha_{m+1} = \frac{(m+1)}{2a_1k}\epsilon^{-1} + \frac{-2(m+1)a_2 + a_1^2(1+m-4A_0k)}{4a_1^2k} + O(\epsilon),$$

$$\alpha_{m+2} = -1 + \frac{(m+3)a_1}{m+1}\epsilon + \frac{(3+4m+m^2)a_2 - (m+2)a_1^2(m+1-4A_0k)}{(m+1)^2}\epsilon^2 + O(\epsilon^3),$$

$$\alpha_{m+3} = -\frac{m+2+(m+1)A_0k}{(m+3)k} + O(\epsilon).$$

This means that α_n is confined in $n = m+3$.

5.2. **Lower triangular matrices with N=2.** The coefficients of equations (61) and (62) are written as

$$\alpha_{m-1,i} = \begin{pmatrix} A_i & 0 \\ C_i & D_i, \end{pmatrix} \quad i = 0, ..., \infty, \tag{67}$$

$$\alpha_{m,i} = \begin{pmatrix} a_i & 0 \\ c_i & d_i, \end{pmatrix}, \quad i = 0, ..., \infty. \tag{68}$$

If we substitute (61) and (62) in (**??**), taking into account (67) and (68), we have that $\alpha_{m+1} = O(\epsilon^{-1})$, and this singularity not necessarily disappear for $n > m+1$. But there are some cases in which it disappear (i.e. the singularity is confined), as the four we will study below. It can be shown that the abelian cases (the ones in which α_{m-1}, α_m and k commute with each other)are particular cases of the ones we will study below.

If we want (62) to satisfy (69), we have that either $a_0 = \pm 1$, or $d_0 = \pm 1$, or both of these cases are valid. Then we will study four cases: $a_0 = 1$, $d_0 = -1$, and $a_0 = -1$ (either with $d_0 = 1$ or $d_0 = -1$).

Then equation (63) becomes

$$\det(\mathbb{I}_N-\alpha_m^2)=\det(\mathbb{I}_N-\alpha_{m,0}^2+O(\epsilon))=\delta_{m,i}\epsilon^s+O(\epsilon^{s+1}), \tag{69}$$

where the i denotes the case we are studying (i=1,...,4).

5.2.1. $a_0=1$. In this case, if we substitute (62) in (69), we get

$$\delta_{m,1}=\frac{2a_1c_0(-2k_3+c_0(k_1-k_4))(k_1-k_4)}{k_3^2}.$$

If we substitute (61) and (62) in (**??**), taking into account (67) and (68), we have that

$$\alpha_{m+1}=K_{-1,1}\epsilon^{-1}+K_{0,1}+K_{1,1}\epsilon+O(\epsilon^2), \tag{70}$$

where

$$K_{-1,1}=\begin{pmatrix}\frac{m+1}{2a_1k_1} & 0\\ K_{-1,1,21} & 0\end{pmatrix},$$

so

$$\det(\mathbb{I}_N-\alpha_{m+1}^2)=\delta_{m+1,1}\epsilon^{-2}+O(\epsilon^{-1}). \tag{71}$$

If we substitute (62) and (70) in (**??**) we get

$$\alpha_{m+2}=K_{0,2}+K_{1,2}\epsilon+O(\epsilon^2), \tag{72}$$

$$K_{0,2}=\begin{pmatrix}-1 & 0\\ K_{0,2,21} & K_{0,2,22}\end{pmatrix},$$

and then

$$\det(\mathbb{I}_N-\alpha_{m+2}^2)=\delta_{m+2,1}\epsilon+O(\epsilon^2). \tag{73}$$

Finally, substituting (70) and (72) in eq. (**??**) we have that

$$\alpha_{m+3}=K_{0,3}+K_{1,3}\epsilon+O(\epsilon^2), \tag{74}$$

with

$$\det(\mathbb{I}_N-\alpha_{m+3}^2)=\delta_{m+3,1}+O(\epsilon). \tag{75}$$

5.2.2. $d_0 = -1$. If we substitute (62) in (69), we get

$$\delta_{m,2} = -2(-1 + a_0^2)d_1.$$

If we substitute (61) and (62) in (**??**), taking into account (67) and (68), we have that

$$\alpha_{m+1} = L_{-1,1}\epsilon^{-1} + L_{0,1} + L_{1,1}\epsilon + O(\epsilon^2), \tag{76}$$

with

$$L_{-1,1} = \begin{pmatrix} 0 & 0 \\ L_{-1,1,21} & \frac{m+1}{2d_1k_4} \end{pmatrix},$$

so

$$\det(\mathbb{I}_N - \alpha_{m+1}^2) = \delta_{m+1,2}\epsilon^{-2} + O(\epsilon^{-1}). \tag{77}$$

If we substitute (62) and (76) in (**??**) we get

$$\alpha_{m+2} = L_{0,2} + L_{1,2}\epsilon + O(\epsilon^2), \tag{78}$$

$$L_{0,2} = \begin{pmatrix} L_{0,2,11} & 0 \\ L_{0,2,21} & 1 \end{pmatrix},$$

and then

$$\det(\mathbb{I}_N - \alpha_{m+2}^2) = \delta_{m+2,2}\epsilon + O(\epsilon^2). \tag{79}$$

Finally, substituting (76) and (78) in (**??**) we have that

$$\alpha_{m+3} = L_{0,3} + L_{1,3}\epsilon + O(\epsilon^2), \tag{80}$$

with

$$\det(\mathbb{I}_N - \alpha_{m+3}^2) = \delta_{m+3,2} + O(\epsilon). \tag{81}$$

Note that in this case, α_{m+i} and $\det(\mathbb{I}_N - \alpha_{m+i}^2)$ (i=1,2,3) are given by the same equations as in the former case, but with different coefficients.

5.2.3. $a_0 = -1,\ d_0 = 1.$ In this case, if we substitute (62) in (69), we get

$$\delta_{m,3} = -4a_1 d_1.$$

If we substitute (61) and (62) into eq. (**??**), taking into account (67) and (68), we have that

$$\alpha_{m+1} = P_{-1,1}\epsilon^{-1} + P_{0,1} + P_{1,1}\epsilon + O(\epsilon^2), \tag{82}$$

with

$$P_{-1,1} = \begin{pmatrix} \frac{m+1}{2a_1k_1} & 0 \\ P_{-1,1,21} & \frac{m+1}{2d_1k_4} \end{pmatrix}.$$

Thus,

$$\det(\mathbb{I}_N - \alpha_{m+1}^2) = \delta_{m+1,3}\epsilon^{-4} + O(\epsilon^{-1}). \tag{83}$$

If we substitute (62) and (82) into eq. (**??**), we get that α_{m+2} is given by

$$\alpha_{m+2} = P_{0,2} + P_{1,2}\epsilon + O(\epsilon^2), \tag{84}$$

where

$$P_{0,2} = \begin{pmatrix} 1 & 0 \\ P_{0,2,21} & -1 \end{pmatrix},$$

and

$$\det(\mathbb{I}_N - \alpha_{m+2}^2) = \delta_{m+2,3}\epsilon^2 + O(\epsilon^3). \tag{85}$$

Thus we have that $\alpha_{m+3} = O(1)$, and

$$\det(\mathbb{I}_N - \alpha_{m+3}^2) = \delta_{m+3,3} + O(\epsilon). \tag{86}$$

5.2.4. $a_0 = -1,\ d_0 = -1.$ We substitute (62) in (69), and we get

$$\delta_{m,4} = 4a_1 d_1$$

(see equation (69)). Now we have that

$$\alpha_{m+1} = Q_{-2,1}\epsilon^{-2} + Q_{-1,1}\epsilon^{-1} + Q_{0,1} + O(\epsilon), \tag{87}$$

with

$$Q_{-2,1} = \begin{pmatrix} 0 & 0 \\ Q_{-2,1,21} & 0 \end{pmatrix},$$

where

$$\det(\mathbb{I}_N - \alpha_{m+1}^2) = \delta_{m+1,4}\epsilon^{-4} + O(\epsilon^{-3}). \tag{88}$$

Now, α_{m+2} satisfies

$$\alpha_{m+2} = Q_{0,2} + Q_{1,2}\epsilon + O(\epsilon^2), \tag{89}$$

where

$$Q_{0,2} = \begin{pmatrix} 1 & 0 \\ Q_{0,2,21} & 1 \end{pmatrix},$$

so

$$\det(\mathbb{I}_N - \alpha_{m+2}^2) = \delta_{m+2,4}\epsilon^{2} + O(\epsilon^{-3}). \tag{90}$$

Finally, we have that $\alpha_{m+3} = O(1)$ and that

$$\det(\mathbb{I}_N - \alpha_{m+3}^2) = \delta_{m+3,4} + O(\epsilon). \tag{91}$$

6. Appendix

6.1. Relations concerning the polynomials $P_j^L(z)$ and $P_j^R(z)$. .

Lemma 1. *If*

$$[k, P_j^L(z)] = 0, \;\; j = 0, ..., m, \tag{92}$$

for some $m \geq 0$*, then*

$$[k, P_n^L(z)] = 0, \;\; n = 0, ..., \infty. \tag{93}$$

Proof. If we calculate the conjugate transpose of the first and the second member of equation (1), we have that

$$\int_{\mathbb{T}} z^j d\mu [P_n^L(z)]^* = -\int_{\mathbb{T}} d\mu [P_n^L(\bar{z})]^* z^{-j} = 0, \;\; j = 0, \ldots, n-1, \tag{94}$$

where we have used the fact that $w(z)$ is hermitian. If we make a comparison between the second member of this equation and equation (2), we have that

$$P_n^R(z) = [P_n^L(\bar{z})]^*. \tag{95}$$

Now, suppose that (92) is valid for some m. Then from (95) we also have that

$$[k, \tilde{P}_j^R(z)] = 0, \;\; j = 0, ..., m, \tag{96}$$

From equations (8), (13) we can see that $[k, b_{j,1}] = [k, \gamma_j] = 0$, so from equation (31) we have that

$$[k, \alpha_j] = 0 \;\;, j = 0, ..., m. \tag{97}$$

If we take into account (92), (96) and (97), it follows from (30) that $[k, P_{m+1}^L(z)] = 0$, so that

$$[k, P_j^L(z)] = 0, \;\; j = 0, ..., m+1. \tag{98}$$

This equation and (92) are valid for $m = 0$. This implies (93).

□

6.2. **Proof of Hermiticity of the matrices α_n.**

Lemma 2. *The relations*

$$\alpha_n = \alpha_n^* \tag{99}$$

hold.

Proof. From the definitions of γ_n (7) and (8), and from the fact that γ_n is hermitian, we have that

$$-2\pi\gamma_n^{-1} = \int_{\mathbb{T}} z^{-n} P_n^L(z) d\mu = \int_{\mathbb{T}} z^{-n} P_n^R(z) d\mu.$$

From (93) we have that $[P_n^L(z), w(z)] = 0$. This means that from (1) and (2) we have that

$$P_n^R(z) = P_n^L(z) := P_n(z). \tag{100}$$

If we use again (1) and (2) we obtain that

$$[P_n(z), w(z)] = [\tilde{P}_n(z), w(z)] = 0. \tag{101}$$

Then, from (95) and (100) we have that $P_n(z)$ has hermitian coefficients. From equations (30) and (100) we have that $P_{n+1}(0) = -\alpha_n$. This means that (99) is valid.

□

6.3. **Proof of the commutativity of the Algebra** (34)-(35).

Lemma 3. *Proof of eq. (34).*

Proof. Suppose that for an integer $m \geq 0$ we have that

$$[\alpha_n, \alpha_{n+i}] = 0,\ i = -n, ..., 0, ..., m. \tag{102}$$

Then from (33) we have that $[P_{n+i}(z), \alpha_n] = [\tilde{P}_{n+i}(z), \alpha_n] = 0$. Using again (33) we have that

$$[P_{n+i+1}(z), \alpha_n] = 0.$$

From (31), the second and the third member of (35), and the definitions of γ_n and $b_{n,1}$ we have that $[\alpha_{n+i+1}, \alpha_n] = 0$. Then we have that

$$[\alpha_{n+i}, \alpha_n] = 0,\ i = -n, ..., 0, ..., \infty.$$

This equation is valid because (102) is valid for all $n \in \mathbb{Z}$ and $m = 0$.

□

Lemma 4. *Proof of eq. (35).*

Proof. If we take into account equations (8) and (13), from (93) and (101) we have that $[k, \gamma_n] = 0$ and $[k, b_{n,1}] = 0$, respectively. Then, from (31) we have that $[k, \alpha_n] = 0$. □

Lemma 5. *Proof of eq. (36).*

Proof. If we apply (34) to (33) we have that

$$[P_m(z), P_n(z')] = 0,\ m, n \geq 0,\ z, z' \in \mathbb{C}, \tag{103}$$

$$[P_m(z), \alpha_n] = 0,\ m, n \geq 0,\ z \in \mathbb{C}. \tag{104}$$

If we use (8) we have from (103), (104), (93) and (35) that (36) is valid, respectively.

□

6.4. **Derivation of the equation** (**??**).

$$\begin{gathered}-\alpha_n^2\alpha_{n-1}^2 - k\alpha_{n+1}\alpha_n\alpha_{n-1}^2(1-\alpha_n^2) - (n-1)\alpha_n\alpha_{n-2}(1-\alpha_{n-1}^2) + \\ k\alpha_n^2\alpha_{n-1}\alpha_{n-2}(1-\alpha_{n-1}^2) - k\alpha_n\alpha_{n-3}(1-\alpha_{n-1}^2)(1-\alpha_{n-2}^2) + k\alpha_n\alpha_{n-2}^2\alpha_{n-1}(1-\alpha_{n-1}^2) + \\ k\alpha_{n-1}\alpha_{n+2}(1-\alpha_n^2)(1-\alpha_{n+1}^2) + (n+2)\alpha_{n-1}\alpha_{n+1}(1-\alpha_n^2) - k\alpha_n\alpha_{n-1}\alpha_{n+1}^2(1-\alpha_n^2) = 0,\ n \geq 3,\end{gathered} \tag{105}$$

with

$$\begin{gathered}-\alpha_0^2 - k\alpha_2(1-\alpha_0^2)(1-\alpha_1^2) + k\alpha_1^2\alpha_0(1-\alpha_0^2) - \\ 2\alpha_1(1-\alpha_0^2) - k\alpha_0\alpha_1(1-\alpha_0^2) = 0,\ n = 0,\end{gathered} \tag{106}$$

$$
\begin{aligned}
-\alpha_1^2\alpha_0^2+k\alpha_0(1-\alpha_1^2)[\alpha_3(1-\alpha_2^2)-\alpha_1\alpha_2^2-\alpha_0\alpha_1\alpha_2]+\\
3\alpha_0\alpha_2(1-\alpha_1^2)-k\alpha_1^2\alpha_0(1-\alpha_0^2)+k\alpha_0\alpha_1(1-\alpha_0^2)=0,\ n=1,
\end{aligned}
\tag{107}
$$

and

$$
\begin{aligned}
-\alpha_1^2\alpha_2^2-k\alpha_1^2\alpha_2\alpha_3(1-\alpha_2^2)+k\alpha_1\alpha_4(1-\alpha_2^2)(1-\alpha_3^2)-\\
k\alpha_1\alpha_2\alpha_3^2(1-\alpha_2^2)+4\alpha_1\alpha_3(1-\alpha_2^2)+\\
k\alpha_0\alpha_1\alpha_2^2(1-\alpha_1^2)-\alpha_0\alpha_2(1-\alpha_1^2)+\\
k\alpha_0^2\alpha_1\alpha_2(1-\alpha_1^2)+k\alpha_2(1-\alpha_0^2)(1-\alpha_1^2)=0,\ n=2,
\end{aligned}
\tag{108}
$$

$$
\begin{aligned}
-2\alpha_n(\sum_{i=0}^{n-2}\alpha_{i+1}\alpha_i-\alpha_0)-k\alpha_n-k\alpha_n^2\alpha_{n-2}(1-\alpha_{n-1}^2)-(n+2)\alpha_n^2\alpha_{n-1}-\\
k\alpha_{n+1}^2\alpha_n(1-\alpha_n^2)+k\alpha_n^3\alpha_{n-1}^2-2k\alpha_{n+1}\alpha_{n-1}\alpha_n(1-\alpha_n^2)+\\
k\alpha_{n+2}(1-\alpha_n^2)(1-\alpha_{n+1}^2)+(n+2)\alpha_{n+1}(1-\alpha_n^2)=0,\ n\geq 2,
\end{aligned}
\tag{109}
$$

$$
\begin{aligned}
-k\alpha_0+k\alpha_2(1-\alpha_0^2)(1-\alpha_1^2)-k\alpha_1^2\alpha_0(1-\alpha_0^2)+2\alpha_1(1-\alpha_0^2)+\\
k\alpha_0^3+2\alpha_0^2+2k\alpha_0\alpha_1(1-\alpha_0^2)=0,\ n=0,
\end{aligned}
\tag{110}
$$

and

$$
\begin{aligned}
-2k\alpha_0\alpha_1\alpha_2(1-\alpha_1^2)+2\alpha_1\alpha_0-k\alpha_1+\\
k\alpha_3(1-\alpha_1^2)(1-\alpha_2^2)-k\alpha_1\alpha_2^2(1-\alpha_1^2)-3\alpha_1^2\alpha_0+\\
3\alpha_2(1-\alpha_1^2)+k\alpha_0^2\alpha_1^3+k\alpha_1^2(1-\alpha_0^2)=0,\ n=1,
\end{aligned}
\tag{111}
$$

$$
\begin{aligned}
k\alpha_n^2\alpha_{n-1}^3-2k\alpha_n\alpha_{n-1}\alpha_{n-2}(1-\alpha_{n-1}^2)-\\
k\alpha_{n+1}\alpha_{n-1}^2(1-\alpha_n^2)-(n+1)\alpha_{n-1}^2\alpha_n-\\
2\alpha_{n-1}(\sum_{i=0}^{n-2}\alpha_{i+1}\alpha_i-\alpha_0)-k\alpha_{n-1}+(n-1)\alpha_{n-2}(1-\alpha_{n-1}^2)-\\
k\alpha_{n-1}\alpha_{n-2}^2(1-\alpha_{n-1}^2)+k\alpha_{n-3}(1-\alpha_{n-1}^2)(1-\alpha_{n-2}^2)=0,\ n\geq 3,
\end{aligned}
\tag{112}
$$

with

$$
-k(1-\alpha_0^2)\alpha_1+k-(1+k\alpha_0)\alpha_0=0,\ n=0,
\tag{113}
$$

$$-2\alpha_0^2\alpha_1 + 2k\alpha_1\alpha_0(1-\alpha_0^2) + k\alpha_1^2\alpha_0^3 - k\alpha_2\alpha_0^2(1-\alpha_1^2) + 2\alpha_0^2 - k\alpha_0 - k\alpha_0(1-\alpha_0^2) = 0,\ n=1, \tag{114}$$

and

$$\begin{gathered}-3\alpha_2\alpha_1^2 + k\alpha_2^2\alpha_1^3 - 2k\alpha_2\alpha_1\alpha_0(1-\alpha_1^2) - \\ k\alpha_3\alpha_1^2(1-\alpha_2^2) - 2\alpha_1^2\alpha_0 + \alpha_0(1-\alpha_1^2) + 2\alpha_1\alpha_0 - \\ k\alpha_1 - k\alpha_1\alpha_0^2(1-\alpha_1^2) - k(1-\alpha_0^2)(1-\alpha_1^2) = 0,\ n=2.\end{gathered} \tag{115}$$

Equations (105)-(108) come from (56), equations (109)-(111) come from (57), and equations (112)-(115) come from (58). Equation (59) trivializes.

We can observe that equations (105), (109) and (112) are equivalent if their initial conditions are satisfied. If we multiply equation (109) by α_{n-1} and equation (112) by α_n we have that both of them can be written in this way:

$$-2\alpha_n\alpha_{n-1}\Big(\sum_{i=0}^{n-2}\alpha_{i+1}\alpha_i - \alpha_0\Big) = F(n,k,\alpha_{n+2},\alpha_{n+1},\alpha_n,\alpha_{n-1},\alpha_{n-2},\alpha_{n-3}). \tag{116}$$

Taking into account this fact, if we write equation (109) and (112) as (116) and we subtract it one from each other, we obtain equation (105). Note that in a similar way we can obtain initial conditions of equation (105) (Eq. (106)-(108)) from initial conditions of equations (109) (Eq. (110)-(111)), and initial conditions of equation (112) (Eq. (113)-(115)). This means that either equations (109)-(111) or equations (112)-(115) are redundant.

By combining algebraically the previous equations (105)-(115), after some straightforward (and tedious) manipulations, we are led to the matrix recurrence (**??**).

Acknowledgment

G. C.-C. benefitted of the financial support of a “Acción Especial” Ref. AE1/13-18837 of the Universidad Complutense de Madrid. M. M. B. thanks economical support from the Spanish “Ministerio de Economía y Competitividad” research project MTM2012-36732-C03-01, *Ortogonalidad y aproximacion; Teoria y Aplicaciones* The research of P. T. has been supported by the grant FIS2011–00260, Ministerio de Ciencia e Innovación, Spain.

References

[1] M. J. Ablowitz, R. Halburd, B. Herbst, *On the extension of the Painleveproperty to difference equations.* Nonlinearity **13**, 889-905 (2000)

[2] M. Adler, P. van Moerbeke, P. Vanhaecke, *Singularity confinement for a class of m-th order difference equations of combinatorics.* Philosophical Transactions of the Royal Society of London A **366**, 877-922 (2008).

[3] W. Van Assche, *Discrete Painlevé equations for recurrence coeffcients of orthogonal polynomials*, Proceedings of the International Conference on Difference Equations, special Functions and Orthogonal Polynomials, World Scientfic (2007), 687-725.

[4] A. I. Bobenko and Y. Suris, *Discrete differential geometry. Integrable structure*, Graduate Studies in Mathematics, **98**. American Mathematical Society, Providence, RI, xxiv+404 pp. (2008).

[5] M. P. Bellon, C.-M. Viallet, *Algebraic entropy*, Communications in Mathematical Physics **204**, 425-437 (1999).

[6] G. Cassatella Contra, M. Mañas Baena, *Riemann–Hilbert problems, matrix orthogonal polynomals and discrete matrix equations with singularity confinement*, Stud. Appl. Math, Vol. 128, Issue 3, pp. 252-274 (2012).

[7] G. Cassatella Contra, M. Mañas Baena, P. Tempesta, *Singularity confinement for matrix discrete Painlevé equations*, Nonlinearity, **27**, 2321-2335 (2014).

[8] R. Conte (Editor), *The Painlevé Property. One Century Later*, Springer–Verlag, New York (1999).

[9] A. S. Fokas, A. R. Its and A. V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, Commun. Math. Phys. **147** (1992), 395-430.

[10] B. Grammaticos, A. Ramani and V. Papageorgiou, *Do integrable mappings have the Painlevé property?* Phys. Rev. Lett **67**, 1825–1828 (1991).

[11] J. Hietarinta and C. Viallet, Physical Review Letters **81**, (1998) 325-328.

[12] S. Lafortune, A. Ramani, B. Grammaticos, Y. Ohta, K. M. Tamizhmani, *Blending two discrete integrability criteria: singularity confinement and algebraic entropy.* Bäcklund and Darboux transformations. The geometry of solitons (Halifax, NS, 1999), 29911, CRM Proceedings and Lecture Notes, 29, Amererican Mathematical Society, Providence, RI, 2001.

[13] A. P. Magnus, *Freud's equations for orthogonal polynomials as discrete Painlevé equations*, Symmetries and Integrability of Difference Equations (Canterbury, 1996), London Math. Soc. Lecture Note Ser., 255, Cambridge University Press, 1999, pp. 228-243.

[14] J. Moser and A. P. Veselov, *Discrete versions of some classical integrable systems and factorization of matrix polynomials*, Commun. Math. Phys. **139**, 217–243 (1991).

[15] I. Dynnikov and S. P. Novikov, *Geometry of the triangle equation on two–manifolds*, Mosc. Math. J. **3**, no. 2, 419–438 (2003).

[16] P. Painlevé, *Leçons sur la théorie analytique des équations différentielles (Leçons de Stockholm, delivered in 1895)*, Hermann, Paris (1897). Reprinted in Œuvres de Paul Painlevé, vol. I, Éditions du CNRS, Paris (1973).

[17] A. Ramani, B. Grammaticos, T. Tamizhmani, K. M. Tamizhmani, *The road to the discrete analogue of the Painleveproperty: Nevanlinna meets singularity confinement.* Computers & Mathematics with Applications **45**, 1001012 (2003).

[18] Yu. B. Suris, *The Problem of Integrable Discretization: Hamiltonian Approach*, Progress in Mathematics, Vol. 219. Basel: Birkhauser, 2003.

[19] G. Freud, *On the coefficients in the recursion formulae of orthogonal polynomials*, Proc. Royal Irish Acad. **A76** (1976), 1-6.

Departamento de Física Teórica II (Métodos Matemáticos de la Física), Facultad de Físicas, Universidad Complutense de Madrid, 28040 Madrid, Spain

E-mail address: gaccontra@fis.ucm.es

Departamento de Física Teórica II (Métodos Matemáticos de la Física), Facultad de Físicas, Universidad Complutense de Madrid, 28040 Madrid, Spain

E-mail address: manuel.manas@ucm.es

Departamento de Física Teórica II (Métodos Matemáticos de la Física), Facultad de Físicas, Universidad Complutense de Madrid, 28040 Madrid, Spain and Instituto de Ciencias Matemáticas, C/ Nicolás Cabrera, No 13–15, 28049 Madrid, Spain

E-mail address: p.tempesta@fis.ucm.es

APPENDIX

DISCRETE MULTISCALE ANALYSIS: A BIATOMIC LATTICE SYSTEM

Article

World Scientific
www.worldscientific.com

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics, Vol. 17, No. 3 (2010) 357–377

DOI: 10.1142/S1402925110000957

DISCRETE MULTISCALE ANALYSIS: A BIATOMIC LATTICE SYSTEM

G. A. CASSATELLA CONTRA* and D. LEVI†

*Departamento de Física Teórica II, (Métodos Matemáticos de la Física)
Universidad Complutense de Madrid, Ciudad Universitaria
28040 — Madrid, Spain

†Dipartimento di Ingegneria Elettronica
Università degli Studi Roma Tre and Sezione
INFN Roma Tre, Via della Vasca Navale 84
00146 Roma, Italy

Received 26 October 2009
Accepted 13 January 2010

We discuss a discrete approach to the multiscale reductive perturbative method and apply it to a biatomic chain with a nonlinear interaction between the atoms. This system is important to describe the time evolution of localized solitonic excitations.

We require that also the reduced equation be discrete. To do so coherently we need to discretize the time variable to be able to get asymptotic discrete waves and carry out a discrete multiscale expansion around them. Our resulting nonlinear equation will be a kind of discrete Nonlinear Schrödinger equation. If we make its continuum limit, we obtain the standard Nonlinear Schrödinger differential equation.

Keywords: Multiple scale expansions; asymptotic analysis on the lattice; integrable equations; nonlinear chains; discrete Nonlinear Schrödinger equation; biatomic lattices.

1. Introduction

Nonlinear systems, and in particular nonlinear discrete systems, are gaining an increasing impact in modern science [33].

In 1955 Fermi, Pasta and Ulam (FPU) [14] considered a unidimensional chain of atoms with nonlinear nearest neighboring interaction to verify if nonlinearity could produce energy equipartition. Instead, they found recurrence, i.e. the motion of the chain for small energies was almost periodic [43]. To explain this result Kruskal and Zabusky found in 1965 [42] a connection between the FPU system and the Korteweg–De Vries equation (KdV), an equation introduced in fluid dynamics to describe one dimensional surface waves in the shallow water context [20]. By introducing the Inverse Scattering Transform, they were able to solve the Cauchy problem for the KdV equation [15] and to prove the existence of soliton solutions.

In 1967 Toda [37] considered a dynamical system with exponential interaction, $U(r) = e^{-r} + r - 1$, the "Toda potential", whose small amplitude approximation gives the FPU system, and shares many of the integrability properties of the KdV equation. So the FPU system turns out to be an approximation of a discrete soliton model.

Later more complicate atomic chains have been considered, as, for example, the biatomic one [6, 9, 11, 12, 16, 26]. These systems have various applications in physics and biology as, for example, in the study of ferroelectric perovskites, materials that, in certain crystallographic directions have an almost unidimensional frame, and in organic molecular chains. A biatomic chain of neighboring atoms A_1 and A_2 is described by the discrete independent variable n and a continuous time t. However, the simplest nonlinear coupled lattice dynamical equations one can construct for this system are not solvable. Only special exact solutions may be found.

Multiscale expansions [7, 8, 19–21, 35, 36] have proved to be important tools to find approximate solutions for many physical problems by reducing a given nonlinear partial differential equation to a simpler equation, which is often integrable [5]. Recently, few attempts to carry over this approach to partial difference equations have been proposed [2, 10, 22, 23, 32]. Almost all approaches considered contain some approximation, either based on physical or on mathematical reasoning as scaling transformations of the lattice provide a nonlocal result. In the following we prefer to stick to mathematical approximations as in this case it will be more evident what to do to improve the final result [17].

In [9] a biatomic chain obtained as a first nonlinear approximation of a complex Lenard–Jones interaction between atoms has been considered. There the multiscale expansion of the continuous limit of the lattice model showed that the modulation of periodic solutions is governed by the Nonlinear Schrödinger differential Equation (NLSE). Here we consider the same model but we are interested in carrying out the multiscale expansion on the lattice, i.e. we are looking for a lattice equation which in the asymptotic regime approximate the biatomic nonlinear lattice. To do so we need to discretize time to be able to allow for discrete asymptotic waves. If we keep a continuous time variable an asymptotic wave travelling on the lattice by necessity will be described by a continuous variable. So by necessity we go over to a differential system.

Discretization of variables, besides representing an interesting problem in mathematical physics for its computerizability, it is also useful in itself. Measurements, for example, are based on sampling of physical variables such as space and time. It follows that physical models in which variables are defined on the lattice are easier to be compared with the real world we see in our measurements.

In this work, we propose to continue the previous researches of biatomic chains considering both t and n as discrete variables. In particular, we shall assume, as these authors, that the system has an unharmonic cubic potential as in nature, potentials usually are non–symmetric. We shall thus apply a discrete multiscale reductive perturbative method to the model introduced by Campa *et. al.* [9] consisting of a biatomic chain with a nonlinear nearest neighbor interaction.

In Sec. 2, we describe in detail the biatomic chain and write down the dynamical equations. Then in Sec. 3, we introduce some notions of discrete calculus and multiple scales defined on the lattice which we apply in Sec. 4 to the biatomic chain introduced in Sec. 2.

In Sec. 5, we analyze the resulting nonlinear discrete equation obtained and carry out its continuum limit. Finally, in Sec. 6, we draw some final conclusions.

2. The Model

We want to describe here a chain suitable to represent, for example, an α-helix channel, see Scott (1999) [33]. Our model consists of a biatomic chain formed by a sequence of pairs of neighboring atoms A_1 and A_2, with masses M_1 and M_2, respectively. Each pair, made of an atom of mass M_1 and the following one of mass M_2, can be considered as a "molecule". We denote by the index n the nth molecule formed by the atom A_1 and A_2 (see Fig. 1). Let us indicate with $x_n(t)$ and $y_n(t)$ the displacements of the atoms A_1 and A_2 belonging to the same molecule n. For each atom, we assume only nearest neighboring interactions. Then, the total potential of the chain is given by

$$U = \sum_n \{U_1(y_n - x_n) + U_2(x_{n+1} - y_n)\}, \tag{1}$$

where U_1 is the intramolecular potential, between atoms belonging to the same molecule, and U_2 is the potential between different molecules.

Given a natural [3, 6, 38] asymmetric potential with an absolute minimum in the equilibrium position as, for example, a Lenard–Jones potential, by taking the first terms of its Taylor expansion around the equilibrium position we can write the potentials U_1 and U_2 as

$$U_1(r) = \frac{1}{2}k_1 r^2 + \frac{\epsilon}{3}\beta_1 r^3, \quad U_2(r) = \frac{1}{2}k_2 r^2 + \frac{\epsilon}{3}\beta_2 r^3,$$

where k_1 and k_2 are the harmonic constants, β_1 and β_2 are the cubic interaction constants and ϵ is a small parameter which will play the role of the perturbative parameter. We assume that the interaction between atoms of the same site is stronger than that of atoms of different sites; thus $k_1 > k_2$ and $|\beta_1| > |\beta_2|$. So, the Hamiltonian of our molecular chain turns out to be

$$H = \sum_n \left\{ \frac{1}{2}[M_1 \dot{x}_n^2 + M_2 \dot{y}_n^2] + \frac{1}{2}[k_1(y_n - x_n)^2 + k_2(x_{n+1} - y_n)^2] \right.$$
$$\left. + \frac{\epsilon}{3}[\beta_1(y_n - x_n)^3 + \beta_2(x_{n+1} - y_n)^3] \right\},$$

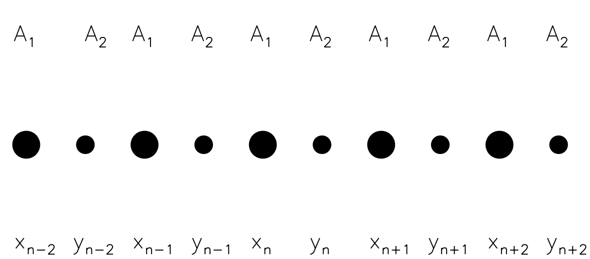

Fig. 1. Pattern of a biatomic molecular chain in one dimension. The chain is formed by a sequence of pairs of neighboring atoms A_1 and A_2. The displacements of the atoms of the molecule n are indicated with x_n and y_n.

where $\dot{x}(t)\equiv\frac{\mathrm{d}x(t)}{\mathrm{d}t}$ and the equations of motion are

$$\begin{aligned} M_1\ddot{x}_n &= -\frac{\partial H}{\partial x_n} \\ &= k_1(y_n - x_n) - k_2(x_n - y_{n-1}) + \epsilon\beta_1(y_n - x_n)^2 - \epsilon\beta_2(x_n - y_{n-1})^2, \end{aligned} \tag{2}$$

$$\begin{aligned} M_2\ddot{y}_n &= \frac{\partial H}{\partial y_n} \\ &= -k_1(y_n - x_n) + k_2(x_{n+1} - y_n) - \epsilon\beta_1(y_n - x_n)^2 + \epsilon\beta_2(x_{n+1} - y_n)^2. \end{aligned} \tag{3}$$

Equations (2) and (3) are a natural extension of the FPU model [14] to a biatomic system.

3. Multiple Scales on a Lattice

Here we introduce the concepts necessary to extend the multiscale reductive perturbative approach introduced by Poincaré [5] for the study of the asymptotic expansion of ordinary differential equations and extended by Taniuti to the reduction of partial differential equations [35, 36] to the case of difference equations [17, 24, 32].

3.1. *Lattices and functions defined on them*

Given a lattice, we will denote by n the running index of the points separated by a constant spacing h. Thus to the lattice **index** n, we can associate a **continuous variable** $x = nh$ defining the position of the points with respect to the origin, for convenience chosen to be with no loss of generality $x_0 = 0$.

If we introduce a small parameter $\epsilon = N^{-1}$, where N is a large integer positive number, we can define on the same lattice the slowly varying discrete variables $n_j (j = 1, 2, 3, \ldots)$ of constant spacing H_j and the continuous variables x_j (see Fig. 2) where

$$n = N^j n_j, \quad x_j = \epsilon^j x. \tag{4}$$

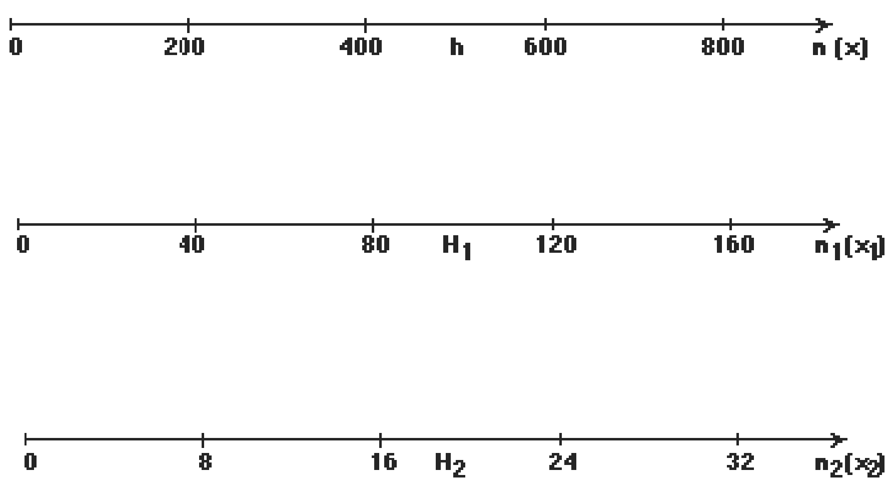

Fig. 2. Rescaled lattices.

If n_j varies by one, n varies by N^j, a number much larger than unity. For this reason, n_j is a "slow variable" and provide an asymptotic behavior of the system. For each j there is a slow lattice variable corresponding to the slow index n_j. n_j will be an integer only if n is a common multiple of N^j.

Let us consider F_n, a function of the discrete index n. An equation on the lattice is a functional relation which involves the function F at various lattice points, $\{F_{n+\ell}\}$. In the case of the model considered before (2, 3), $\ell = \pm 1$. We are interested to transform the system, defined on a lattice n, to the slowly varying lattices n_j, providing the scales of the asymptotic behavior of the original system. This is equivalent to say that we are interested in transforming the system defined on x to the one with the slowly varying variables x_j. We can consider the function F_n written in terms of the slowly varying lattice variables $\{n_j\}$, with, for example, $j = 1, 2$, $F_n \equiv f_{n_1,n_2}$, and we can carry out the ϵ expansion of the function $F_{n+\ell}$.

Let us consider at the beginning the case of one slowly varying lattice n_1, i.e. $F_n \equiv f_{n_1}$. As the shift operator T_n acting on F_n gives $T_n F_n = F_{n+1}$, we have $F_{n+\ell} = T_n^\ell F_n$. In order to extract the behavior of the function $F_{n+1} = F(x+h)$ on the new scales, let us carry out the Taylor expansion of F_{n+1} in powers of h. In such a case the shift operator can be expressed as an infinite order differential operator with respect to x, i.e.

$$T_n = \exp(h\partial_x) = \sum_{k=0}^{\infty} \frac{(h\partial_x)^k}{k!}. \tag{5}$$

Moreover, if we define a Δ operator as $\Delta_n^{(+)} \equiv (T_n - 1)/h$, we have

$$\partial_x = \frac{\log(1 + h\Delta_n^{(+)})}{h}, \tag{6}$$

and Eq. (5) could be written as

$$T_n = \sum_{k=0}^{\infty} \frac{(\log(1 + h\Delta_n^{(+)}))^k}{k!}. \tag{7}$$

Formulas (6) and (7) are written in terms of $\Delta_n^{(+)}$. However on the lattice we can define an infinite number of different difference operators which in the continuum limit, when h goes to zero, go over to the first order derivative. Among them it is important, as it is self-adjoint, the symmetric shift operators $\Delta_n^{(s)} \equiv \frac{1}{2h}(T_n - T_n^{-1})$. In this case we have

$$\partial_x = \frac{\operatorname{arcsinh}(h\Delta_n^{(s)})}{h}, \quad \to \quad T_n = \sum_{k=0}^{\infty} \frac{(\operatorname{arcsinh}(h\Delta_n^{(s)}))^k}{k!}. \tag{8}$$

Introducing the slowly varying variable x_1 and the corresponding lattice n_1 in Eq. (5), as $\partial_x = \epsilon\partial_{x_1}$, we have

$$T_n^\ell = e^{\ell h \partial_x} = e^{\ell\epsilon h\partial_{x_1}} = T_{n_1}^{\ell\epsilon} = \sum_{k=0}^{\infty} \frac{(\ell h\epsilon\partial_{x_1})^k}{k!}. \tag{9}$$

If we introduce more lattice variables, for example $\{n_j\}$, with $j = 1, 2$, then T_n becomes

$$T_n^\ell = T_{n_1}^{\ell\epsilon} T_{n_2}^{\ell\epsilon^2} = \sum_{k=0}^{\infty} \frac{(\ell\epsilon h \partial_{x_1})^k}{k!} \sum_{j=0}^{\infty} \frac{(\ell\epsilon^2 h \partial_{x_2})^j}{j!}. \tag{10}$$

Once we expand the operator ∂_{x_j} in terms of shift operators we get an expression for $F(n \pm \ell)$ in terms of variations of $f(n_1, n_2)$ with coefficients depending on ϵ and ℓ.

As delta operators are linear combinations of shift operators, from Eq. (13) it can be proved [18, 24] that for $\Delta = \Delta^{(+)}$ we have the following formula

$$(\Delta_{n_1}^{(+)})^k f_{n_1} = \sum_{i=k}^{\infty} \frac{k!}{i!} P(i,k) (\Delta_n^{(+)})^i F_n, \tag{11}$$

where $(\Delta_{n_1}^{(+)})^k f_{n_1}$ is the kth-difference of f_{n_1} respect to n_1, and the coefficients $P(i,k)$ are given by $P(i,j) = \sum_{\alpha=j}^{i} w^\alpha S_i^\alpha G_\alpha^j$, where ω is the ratio of the increment in the lattice of variable n with respect to that of variable n_1. In this case, taking into account Eq. (4), $\omega = N$. The coefficients S_i^α and G_α^j are the Stirling coefficients of the first kind and second kind, respectively. The result (11) can be inverted, providing:

$$(\Delta_n^{(+)})^k F_n = \sum_{i=k}^{\infty} \frac{k!}{i!} Q(i,k) (\Delta_{n_1}^{(+)})^i f_{n_1}, \tag{12}$$

where $Q(i,j)$ is the same as $P(i,j)$, but with $w = N^{-1} = \epsilon$.

A general way to get these formulas is provided by the *finite operator calculus* [13, 29, 30]. The finite operator calculus prescribes the following formula [25]

$$T_n^j = \sum_{k=0}^{\infty} \frac{(\epsilon)^k p_k(j)}{k!} (\Delta_{n_1})^k, \tag{13}$$

where the functions $p_k(j)$ are the unique basic sequence associated to the operator Δ_{n_1}, i.e. such that they satisfy the following conditions

$$\begin{gathered} p_0(n_1) = 1, \quad p_k(0) = 0 \quad \text{for all } k > 0, \\ \Delta_{n_1} p_k(n_1) = k p_{k-1}(n_1). \end{gathered} \tag{14}$$

The basic sequences can be directly obtained by the transfer formulae:

$$p_k(n_1) = n_1 \left(\frac{\Delta_{n_1}}{h \partial_{x_1}} \right)^{-k} {n_1}^{k-1}. \tag{15}$$

When $\Delta_{n_1} = \Delta_{n_1}^{(+)}$ or $\Delta_{n_1} = \Delta_{n_1}^{(s)}$, the basic sequences are:

$$\begin{aligned} p_k^{(+)}(n_1) &= h^k n_1 \left(\frac{e^{h\partial_{x_1}} - 1}{h \partial_{x_1}} \right)^{-k} {n_1}^{k-1} = (x_1)_k \equiv x_1(x_1 - h) \cdots (x_1 - kh + h), \\ p_k^{(s)}(n_1) &= h^k n_1 \left(\frac{e^{h\partial_{x_1}} - e^{-h\partial_{x_1}}}{2h \partial_{x_1}} \right)^{-k} {n_1}^{k-1} = 2^k G_k(x_1; -h, 2h), \end{aligned} \tag{16}$$

where $G_k(y; a, b)$ are the Gould polynomials [29] given by

$$\begin{aligned} G_k(y; a, b) &\equiv \frac{y}{y - ka}\left(\frac{y - ka}{b}\right)_k \\ &= \frac{y}{(y - ka)(b)^k}(y - ka)(y - ka - b)\cdots(y - ka - (k - 1)b). \end{aligned} \tag{17}$$

Let us also mention that for each Δ_{n_1} operator we can write from Eq. (13)

$$(\partial_{x_1})^j = \frac{1}{h^j}\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\mathrm{d}^j}{\mathrm{d}y^j}p_k(y)\right]\Bigg|_{y=0}(\Delta_{n_1})^k, \tag{18}$$

i.e. we can express the partial derivative as an infinite sum of differences whose coefficients depends from the type of difference we are expanding into. In terms of $\Delta^{(+)}$, from Eqs. (13) and (16), Eq. (9) reads

$$T_n^{\ell}F_n = \sum_{k=0}^{\infty}\frac{(h\epsilon)^k(\ell)_k}{k!}(\Delta_{n_1}^{(+)})^k f_{n_1}, \tag{19}$$

while, in the symmetric difference case, it reads

$$T_n^{\ell}F_n = \sum_{k=0}^{\infty}\frac{(2h\epsilon)^k}{k!}G_k(l; -1, 2)(\Delta_{n_1}^{(s)})^k f_{n_1}. \tag{20}$$

From Eqs. (19) and (20) we get that any finite shift in the original equation will give rise to an expression in the slowly varying variables which involves an infinity of lattice points or, equivalently, contains differences at all orders of the function f_{n_1}. So to get a reduced equation on a finite number of points we need to cut the series by requiring that the function f_{n_1} be of finite order of variation. Let us introduce the following definition:

Definition. The function f_n is a slow varying function of order p if

$$\Delta^{p+1}f_n = 0. \tag{21}$$

Then we can prove the following Theorem:

Theorem. *The function F_n is a slow varying function of order p iff f_{n_1} is a slowly varying function of order p in its own variable*, i.e. *if* $\Delta_{n_1}^{p+1}f_{n_1} = 0$.

Proof. The proof of this theorem will be given in the case of $\Delta = \Delta^+$, but it is easy to see that it is valid for any delta operator. It is divided into two parts:

(a) Let f_{n_1} be a slowly varying function of order p. From formula (12) it follows that

$$\Delta_n^{p+1}F_n = \sum_{i=p+1}^{\infty}\frac{(p+1)!}{i!}Q(i, p+1)\Delta_{n_1}^i f_{n_1} = 0, \tag{22}$$

i.e. F_n is also a slow function of order p.

(b) Let F_n be a slowly varying function of order p. From formula (11) it follows that

$$\Delta_{n_1}^{p+1} f_{n_1} = \sum_{i=p+1}^{\infty} \frac{(p+1)!}{i!} P(i, p+1) \Delta_n^i F_n = 0, \tag{23}$$

i.e. f_{n_1} is also a slow function of order p. □

The expansion (20) can be performed in two steps: at first we write the shift operator in the n variable in terms of the derivatives with respect to x_1 by formula (9) and then we expand the derivatives with respect to x_1 in term of delta operators by formula (18). In doing so we will have formulas in derivatives which are valid for any delta operator. Moreover the first expansion has ϵ dependent coefficients while the second will provide a finite number of terms only if we use the slow varying condition for the functions f_{n_1,n_2}.

Let us now explicitate the first terms of Eq. (20) for future use, at first in terms of the derivatives and then in delta operators assuming that the function f_{n_1,n_2} is a slow function at most of order 2. At first we shall consider the case in which we have only one slow lattice, just the variable n_1 is present and then we extend the result to the case of two slow lattices, n_1 and n_2 and to partial lattices n and m.

3.1.1. $F_n = f_{n_1} = f(x_1)$

From Eq. (9) we get

$$F_{n\pm 1} = f_{(x_1)} \pm h\epsilon \partial_{x_1} f(x_1) + \frac{h\epsilon^2}{2!} \partial_{x_1}^2 f(x_1) + \mathcal{O}(\epsilon^3). \tag{24}$$

As from Eq. (18) for $p = 2$, $\partial_{x_1} = \Delta_{n_1}$ and $\partial^2{}_{x_1} = (\Delta_{n_1})^2$, then Eq. (24) reads

$$F_{n\pm 1} = f_{n_1} \pm \frac{1}{2N}(f_{n_1+1} - f_{n_1-1}) + \frac{1}{2N^2}(f_{n_1+1} - 2f_{n_1} + f_{n_1-1}) + O(N^{-3}). \tag{25}$$

3.1.2. $F_n = f_{n_1,n_2} = f(x_1, x_2)$

$p = 2$ is the lowest nontrivial value of p for which we can consider F_n as a function of the two scales, n_1 and n_2. Taking $l = 1$, from Eq. (10) we have

$$\begin{aligned} F_{n\pm 1} = f(x_1, x_2) &\pm h\epsilon \frac{\partial f(x_1, x_2)}{\partial x_1} + \frac{h^2\epsilon^2}{2} \frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} \pm h\epsilon^2 \frac{\partial f(x_1, x_2)}{\partial x_2} \\ &+ h^2\epsilon^3 \frac{\partial}{\partial x_1} \frac{\partial f(x_1, x_2)}{\partial x_2} + O(\epsilon^4). \end{aligned} \tag{26}$$

If F_n is a slowly varying function of order two in n_1, it might be of order one in n_2. In this case, Eq. (26) becomes

$$F_{n\pm 1} = f(x_1, x_2) \pm h\epsilon \frac{\partial f(x_1, x_2)}{\partial x_1} + \frac{h^2\epsilon^2}{2} \frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} \pm h\epsilon^2 \frac{\partial f(x_1, x_2)}{\partial x_2} + O(\epsilon^3). \tag{27}$$

Moreover, from Eq. (18) it follows that $\partial_{x_2} = \Delta_{n_2}, \partial_{x_1}^2 = (\Delta_{n_1})^2$ and $\partial_{x_1}\partial_{x_2} = \Delta_{n_1}\Delta_{n_2}$. Then Eqs. (26) and (27), written in terms of differences instead of derivatives, are given by

$$\begin{aligned} F_{n\pm 1} &= f_{n_1,n_2} \pm \frac{1}{2N}(f_{n_1+1,n_2} - f_{n_1-1,n_2}) \\ &\quad + \frac{1}{2N^2}(f_{n_1+1,n_2} - 2f_{n_1,n_2} + f_{n_1-1,n_2}) \pm \frac{1}{2N^2}(f_{n_1,n_2+1} - f_{n_1,n_2-1}) \\ &\quad + \frac{1}{4N^3}(f_{n_1+1,n_2+1} - f_{n_1-1,n_2+1} - f_{n_1+1,n_2-1} + f_{n_1-1,n_2-1}) + O(N^{-4}) \end{aligned} \tag{28}$$

and

$$\begin{aligned} F_{n\pm 1} &= f_{n_1,n_2} \pm \frac{1}{2N}(f_{n_1+1,n_2} - f_{n_1-1,n_2}) \\ &\quad + \frac{1}{2N^2}(f_{n_1+1,n_2} - 2f_{n_1,n_2} + f_{n_1-1,n_2}) \\ &\quad \pm \frac{1}{2N^2}(f_{n_1,n_2+1} - f_{n_1,n_2-1}) + O(N^{-3}), \end{aligned} \tag{29}$$

respectively.

3.1.3. $F_{n,m} = f_{n_1,m_1,m_2} = f(x_1, t_1, t_2)$

In this case we have

$$\begin{aligned} F_{n,m\pm 1} &= f(x_1,t_1,t_2) \pm \tau\epsilon \frac{\partial f(x_1,t_1,t_2)}{\partial t_1} \\ &\quad + \frac{\tau^2\epsilon^2}{2}\frac{\partial^2 f(x_1,t_1,t_2)}{\partial t_1^2} \pm \tau\epsilon^2 \frac{\partial f(x_1,t_1,t_2)}{\partial t_2} + O(\epsilon^3), \end{aligned} \tag{30}$$

and

$$F_{n\pm 1,m} = f(x_1,t_1,t_2) \pm h\epsilon \frac{\partial f(x_1,t_1,t_2)}{\partial x_1} + \frac{h^2\epsilon^2}{2}\frac{\partial^2 f(x_1,t_1,t_2)}{\partial x_1^2} + O(\epsilon^3). \tag{31}$$

In terms of differences, the last two equations are given by

$$\begin{aligned} F_{n,m\pm 1} &= f_{n_1,m_1,m_2} \pm \frac{1}{2N}(f_{n_1,m_1+1,m_2} - f_{n_1,m_1-1,m_2}) \\ &\quad + \frac{1}{2N^2}(f_{n_1,m_1+1,m_2} - 2f_{n_1,m_1,m_2} + f_{n_1,m_1-1,m_2}) \\ &\quad \pm \frac{1}{2N^2}(f_{n_1,m_1,m_2+1} - f_{n_1,m_1,m_2-1}) + O(N^{-3}) \end{aligned} \tag{32}$$

and

$$\begin{aligned} F_{n\pm 1,m} &= f_{n_1,m_1,m_2} \pm \frac{1}{2N}(f_{n_1+1,m_1,m_2} - f_{n_1-1,m_1,m_2}) \\ &\quad + \frac{1}{2N^2}(f_{n_1+1,m_1,m_2} - 2f_{n_1,m_1,m_2} + f_{n_1-1,m_1,m_2}) + O(N^{-3}). \end{aligned} \tag{33}$$

For future use we can further rescale the lattice with some extra parameter by defining $n_1 = \frac{L_1 n}{N}, m_1 = \frac{L_2 m}{N}$ e $m_2 = \frac{m}{N^2}$, where the order 1 parameters L_1 and L_2 are divisors of N

and N^2 respectively if we require that n_1 and n_2 be integer numbers. In this case, Eqs. (30) and (31) become

$$F_{n,m\pm 1} = f(x_1,t_1,t_2) \pm \tau L_2 \epsilon \frac{\partial f(x_1,t_1,t_2)}{\partial t_1} + \frac{\tau^2 L_2^2 \epsilon^2}{2} \frac{\partial^2 f(x_1,t_1,t_2)}{\partial t_1^2} \pm \tau \epsilon^2 \frac{\partial f(x_1,t_1,t_2)}{\partial t_2} + O(\epsilon^3), \tag{34}$$

and

$$F_{n\pm 1,m} = f(x_1,t_1,t_2) \pm h L_1 \epsilon \frac{\partial f(x_1,t_1,t_2)}{\partial x_1} + \frac{h^2 L_1^2 \epsilon^2}{2} \frac{\partial^2 f(x_1,t_1,t_2)}{\partial x_1^2} + O(\epsilon^3). \tag{35}$$

Moreover, from Eq. (34) we have

$$F_{n,m+1} - 2F_{n,m} + F_{n,m-1} = \tau^2 L_2^2 \epsilon^2 \frac{\partial^2 f(x_1,t_1,t_2)}{\partial t_1^2} + O(\epsilon^3). \tag{36}$$

In terms of symmetric difference operators these equations can be written as

$$\begin{aligned} F_{n,m\pm 1} = f_{n_1,m_1,m_2} &\pm \frac{L_2}{2N}(f_{n_1,m_1+1,m_2} - f_{n_1,m_1-1,m_2}) \\ &+ \frac{L_2^2}{2N^2}(f_{n_1,m_1+1,m_2} - 2f_{n_1,m_1,m_2} + f_{n_1,m_1-1,m_2}) \\ &\pm \frac{1}{2N^2}(f_{n_1,m_1,m_2+1} - f_{n_1,m_1,m_2-1}) + O(N^{-3}), \end{aligned} \tag{37}$$

$$\begin{aligned} F_{n\pm 1,m} = f_{n_1,m_1,m_2} &\pm \frac{L_1}{2N}(f_{n_1+1,m_1,m_2} - f_{n_1-1,m_1,m_2}) \\ &+ \frac{L_1^2}{2N^2}(f_{n_1+1,m_1,m_2} - 2f_{n_1,m_1,m_2} + f_{n_1-1,m_1,m_2}) + O(N^{-3}) \end{aligned} \tag{38}$$

and

$$\begin{aligned} F_{n,m+1} - 2F_{n,m} + F_{n,m-1} = \frac{L_2^2}{N^2}&(f_{n_1,m_1+1,m_2} - 2f_{n_1,m_1,m_2} + f_{n_1,m_1-1,m_2}) \\ &+ O(N^{-3}). \end{aligned} \tag{39}$$

The last three equations will be used in the following section to apply the multiscale method to the biatomic lattice model we introduced in Sec. 2.

4. Multiscale Reduction of the Discrete Biatomic System

4.1. *Equations of motion*

In the equations of motion of the biatomic chain (see Eqs. (2) and (3)), the nonlinear terms (proportional to β_1 and β_2) are of order ϵ respect to the remaining terms, and thus we can use perturbative methods to look for approximate solutions of $x_n(t)$ and $y_n(t)$. This has been done in 1993 by Campa *et al.* [9] using the multiscale perturbative method with just the lowest order differential terms. In this way, performing at the same time a multiscale expansion and a continuum limit they were able to reduce the system to the NLSE (69).

Here we discretize time and look for completely discrete equations, i.e. passing from the differential terms in the expansion (see Eqs. (24), (26), (27), (30), (31), (34)–(36)) to difference terms corresponding to the lowest order of slow varyness p, i.e. to Eqs. (25), (28), (29), (32), (33), (37)–(39). To discretize time we replace the time t with a discrete variable m, so that $t \equiv m\tau$, where τ is the temporal scale. Thus, when τ reduces to an infinitesimal quantity and m approaches infinity in such a way that t remains finite we recover the continuous case. We consider the simplest approximation of the second derivative by differences using a central difference so as to get a real dispersive relation. The discretized equations of motion are given by

$$\begin{aligned} m_1(x_{n,m+1} - 2x_{n,m} + x_{n,m-1}) &= k_1(y_{n,m} - x_{n,m}) - k_2(x_{n,m} - y_{n-1,m}) \\ &\quad + \epsilon[\beta_1(y_{n,m} - x_{n,m})^2 - \beta_2(x_{n,m} - y_{n-1,m})^2], \quad (40) \\ m_2(y_{n,m+1} - 2y_{n,m} + y_{n,m-1}) &= -k_1(y_{n,m} - x_{n,m}) + k_2(x_{n+1,m} - y_{n,m}) \\ &\quad - \epsilon[\beta_1(y_{n,m} - x_{n,m})^2 - \beta_2(x_{n+1,m} - y_{n,m})^2], \quad (41) \end{aligned}$$

where $x_{n,m} \equiv x_n(m\tau), y_{n,m} \equiv y_n(m\tau)$ and $m_{1,2} \equiv \frac{M_{1,2}}{\tau^2}$. We are looking for $x_{n,m}$ and $y_{n,m}$ as bounded solutions written as a modulation of the harmonic wave solutions of the linearized equations which one obtains when setting $\epsilon = 0$. The harmonic waves are given by

$$E_{n,m} = e^{i[kn - \omega(k)m]}, \quad (42)$$

with $\omega(k)$ real for any real value of k. The physical reason for choosing harmonic waves is that the atoms of the chain make only small oscillations around their equilibrium position. When we introduce this ansatz into Eqs. (40) and (41), we realize at once that the solution of the nonlinear equations of motion can be represented as a modulated linear combination of harmonic functions.

A solution of the linear part of Eqs. (40) and (41) ($\beta_1 = \beta_2 = 0$), written in terms of the harmonic waves (42), is given by

$$x_{n,m} = AE_{n,m}, \quad y_{n,m} = BE_{n,m},$$

where

$$\frac{B}{A} = r \equiv \frac{k_1 + k_2 + 2m_1(\cos\omega(k) - 1)}{k_1 + k_2 e^{-ik}} = \frac{k_1 + k_2 e^{ik}}{k_1 + k_2 + 2m_2(\cos\omega(k) - 1)}, \quad (43)$$

with the dispersion relation

$$\begin{aligned} \omega(k) = \arccos\Bigg\{1 - \frac{1}{4m_1m_2}\Bigg[&(k_1 + k_2)(m_1 + m_2) \\ &\pm \sqrt{(k_1 + k_2)^2(m_1 + m_2)^2 - 16k_1k_2m_1m_2 sin^2\frac{k}{2}}\Bigg]\Bigg\}. \quad (44) \end{aligned}$$

It can be proved that the term inside the square root of the dispersion relation is always positive, so that the argument of "arccos" is always real.

In Eq. (44), the positive sign corresponds to the optical branch $\omega_{\text{opt}}(k)$, whereas the negative one to the acoustical branch $\omega_{\text{ac}}(k)$. It can be proved that the function $\omega(k)$ is real for all real values of k iff the temporal scale τ satisfies the following inequalities:

$$\tau \leq \sqrt{\frac{4M_1M_2}{(k_1+k_2)(M_1+M_2)}} \equiv \tau_o \tag{45}$$

for the optical branch, and

$$\tau \leq \sqrt{\frac{8M_1M_2}{(k_1+k_2)(M_1+M_2) - \sqrt{(k_1+k_2)^2(M_1+M_2)^2 - 16k_1k_2M_1M_2}}} \equiv \tau_a \tag{46}$$

for the acoustical one. It is easy to show that τ_a is always larger than τ_o. In Figs. 3, 4 we show how $\omega(k)$ varies as a function of τ. We have chosen, following Campa [9], the following numerical values for the parameters, $M_1 = 1$, $M_2 = 1.5$, $k_1 = 1$ and $k_2 = 0.3$, so that $\tau_o \simeq 1.358732$ and $\tau_a \simeq 2.910816$. So the obtained threshold values τ_o and τ_a are consistent with the request that τ, the discretization parameter, be smaller than one.

Let us seek a finite amplitude solution of the nonlinear system (40), (41). To do so, we write $x_{n,m}$ and $y_{n,m}$ in terms of the harmonics of the linearized Eq. (42)

$$x_{n,m} = \sum_{s=0}^{\infty} G_{n,m}^s (E_{n,m})^s + \sum_{s=1}^{\infty} \bar{G}_{n,m}^s (E_{n,m})^{-s}, \tag{47}$$

$$y_{n,m} = \sum_{s=0}^{\infty} H_{n,m}^s (E_{n,m})^s + \sum_{s=1}^{\infty} \bar{H}_{n,m}^s (E_{n,m})^{-s}, \tag{48}$$

where, as the variables $x_{n,m}$ and $y_{n,m}$ are real, $(\bar{G}_{n,m}^s, \bar{H}_{n,m}^s)$ are the complex conjugates of the modulation coefficients $(G_{n,m}^s, H_{n,m}^s)$. We choose $G_{n,m}^s = g_{n_1,m_1,m_2}^s$ and $H_{n,m}^s = h_{n_1,m_1,m_2}^s$ as slowly varying functions of the second order in n_1 and m_1 and of the first order

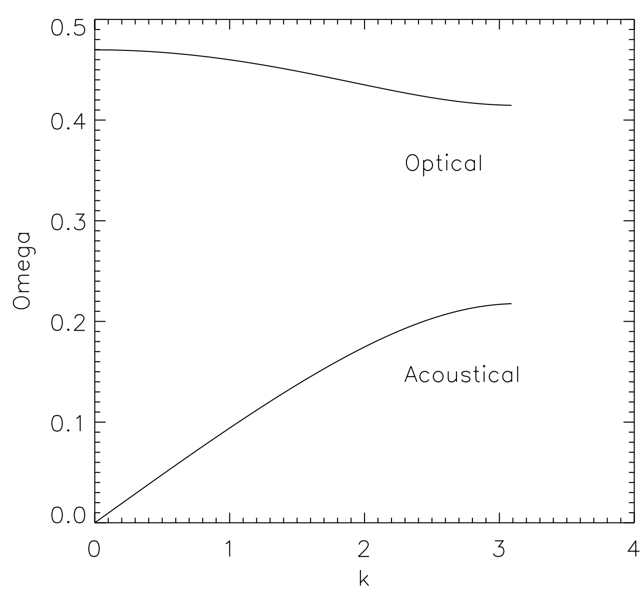

Fig. 3. Graph of $\omega(k)$ against k, with k lying in the interval $[0, \pi]$. We have chosen $M_1 = 1, M_2 = 1.5$, $k_1 = 1$, $k_2 = 0.3$ and $\tau = 10^{-1/2}$.

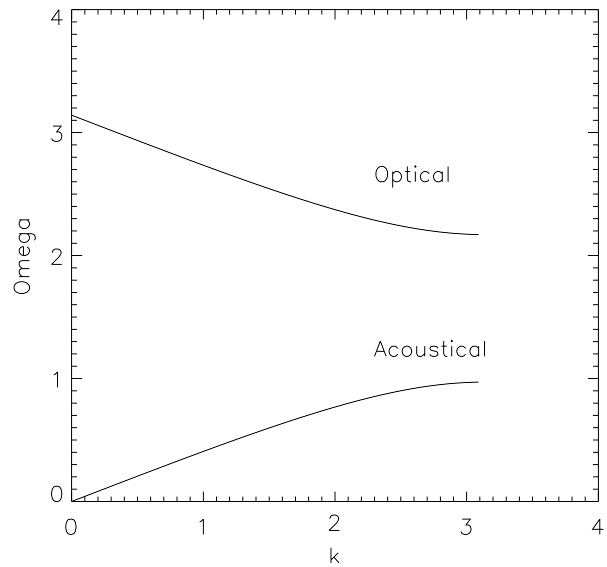

Fig. 4. Graph of $\omega(k)$ against k, with k lying in the interval $[0, \pi]$. The parameters M_1, M_2, k_1, and k_2 are the same as in Fig. 4, but $\tau = \tau_o$.

in m_2, defined in such a way to avoid secular terms. Moreover we expand the functions $g^s_{n_1,m_1,m_2}$ and $h^s_{n_1,m_1,m_2}$ in the small parameter ϵ. So we have:

$$G^s_{n,m} \equiv \sum_{l=0}^{\infty} \epsilon^l g^{(s,l)}_{n_1,m_1,m_2}, \tag{49}$$

$$H^s_{n,m} \equiv \sum_{l=0}^{\infty} \epsilon^l h^{(s,l)}_{n_1,m_1,m_2}. \tag{50}$$

4.2. *Derivation of the equations of motion*

Substituting ansatz (47), (48) into the equations of motion (40), (41) and taking into account Eqs. (49) and (50) we get two equations of the form

$$\sum_{s=0}^{\infty}\sum_{l=0}^{\infty} \epsilon^l F^{(s,l)}_{n_1,m_1,m_2}(E_{n,m})^s + \sum_{s=0}^{\infty}\sum_{l=0}^{\infty} \epsilon^l \bar{F}^{(s,l)}_{n_1,m_1,m_2}(E_{n,m})^{-s} = 0, \tag{51}$$

where the $F^{(s,l)}_{n_1,m_1,m_2}$ are function only of the slow variables. As $(E_{n,m})^s$ and $(E_{n,m})^{-s}$ are independent functions, its coefficients must be equal to zero. So for each power of $(E_{n,m})$ and ϵ we get sets of equations $F^{(s,l)}_{n_1,m_1,m_2} = 0$ for the slow varying modulation coefficients $g^{(s,l)}_{n_1,m_1,m_2}$ and $h^{(s,l)}_{n_1,m_1,m_2}$ together with their complex conjugate.

4.2.1. ϵ^0

We look here for the linearized terms. In this case, the coefficient of the zeroth harmonic satisfies the equation

$$g^{(0,0)}_{n_1,m_1,m_2} = h^{(0,0)}_{n_1,m_1,m_2}, \tag{52}$$

whereas the coefficients of the first harmonics gives a set of two equations that are identically satisfied when $\omega(k)$ satisfies the dispersion relation (44) and

$$\frac{h^{(1,0)}_{n_1,m_1,m_2}}{g^{(1,0)}_{n_1,m_1,m_2}} = r. \tag{53}$$

It can be proven easily that, for $q \geq 2$, $g^{(q,0)}_{n_1,m_1,m_2} = h^{(q,0)}_{n_1,m_1,m_2} = 0$.

4.2.2. ϵ^1

The coefficients of the zeroth harmonic are

$$\begin{aligned} h^{(0,1)}(x_1,t_1,t_2) &= g^{(0,1)}(x_1,t_1,t_2) + \frac{hL_1k_2}{k_1+k_2}\frac{\partial g^{(0,0)}(x_1,t_1,t_2)}{\partial x_1} \\ &\quad + \frac{2}{k_1+k_2}[\beta_2|1-e^{-ik}r|^2 - \beta_1|1-r|^2]|g^{(1,0)}(x_1,t_1,t_2)|^2, \end{aligned} \tag{54}$$

or

$$\begin{aligned} h^{(0,1)}_{n_1,m_1,m_2} &= g^{(0,1)}_{n_1,m_1,m_2} + \frac{L_1k_2}{2(k_1+k_2)}(g^{(0,0)}_{n_1+1,m_1,m_2} - g^{(0,0)}_{n_1-1,m_1,m_2}) \\ &\quad + \frac{2}{k_1+k_2}[\beta_2|1-e^{-ik}r|^2 - \beta_1|1-r|^2]|g^{(1,0)}_{n_1,m_1,m_2}|^2, \end{aligned} \tag{55}$$

depending if we use the expansions in terms of derivatives or differences.

For $s = 1$ we find a system of two equations in the two unknowns, $g^{(1,1)}_{n_1,m_1,m_2}$ and $h^{(1,1)}_{n_1,m_1,m_2}$. This system is compatible only if

$$g^{(1,0)}_{n_1,m_1,m_2} \equiv g^{(1,0)}_{n_2,m_2}, \tag{56}$$

where $n_2 = n_1 - m_1$ and

$$h^{(1,1)}_{n_1,m_1,m_2} = rg^{(1,1)}_{n_1,m_1,m_2} + \frac{2i\sin\omega m_1\omega_{,k} + k_2re^{-ik}}{2(k_1+k_2e^{-ik})}L_1(g^{(1,0)}_{n_2+1,m_2} - g^{(1,0)}_{n_2-1,m_2}), \tag{57}$$

where $\omega_{,k} \equiv \frac{d\omega}{dk} = \frac{L_2}{L_1}$, with L_1 and L_2 given in Appendix A.1 by Eqs. (72) and (73). The differential version of Eq. (56) is

$$g^{(1,0)}(x_1,t_1,t_2) \equiv g^{(1,0)}(x_2,t_2),$$

where $x_2 \equiv hn_2 = h(n_1 - m_1) = x_1 - \frac{h}{\tau}t_1$, and

$$h^{(1,1)}(x_1,t_1,t_2) = rg^{(1,1)}(x_1,t_1,t_2) + \frac{2i\sin\omega m_1\omega_{,k} + k_2re^{-ik}}{k_1+k_2e^{-ik}}hL_1\frac{\partial g(x_2,t_2)^{(1,0)}}{\partial x_2}. \tag{58}$$

For the second harmonic we get

$$g^{(2,1)}_{n_1,m_1,m_2} = K_1g^{(1,0)2}_{n_1,m_1,m_2}, \tag{59}$$

$$h^{(2,1)}_{n_1,m_1,m_2} = K_2g^{(1,0)2}_{n_1,m_1,m_2}, \tag{60}$$

where K_1 and K_2 are given in Appendix A.1 by Eqs. (75) and (76). It can be easily proven that, for $q \geq 3$, $g^{(q,1)}_{n_1,m_1,m_2} = h^{(q,1)}_{n_1,m_1,m_2} = 0$.

4.2.3. ϵ^2

Taking into account Eq. (56), the zeroth harmonic gives a system of two equations that is satisfied only if

$$L_1^2(g^{(0,0)}_{n_2+1,m_2} + g^{(0,0)}_{n_2-1,m_2} - 2g^{(0,0)}_{n_2,m_2}) = L_1\frac{c^0}{2}(|g^{(1,0)}_{n_2+1,m_2}|^2 - |g^{(1,0)}_{n_2-1,m_2}|^2)$$
$$+ L_1\frac{c^1}{2}\{g^{(1,0)}_{n_2,m_2}(\bar g^{(1,0)}_{n_2+1,m_2} - \bar g^{(1,0)}_{n_2-1,m_2}) + \bar g^{(1,0)}_{n_2,m_2}(g^{(1,0)}_{n_2+1,m_2} - g^{(1,0)}_{n_2-1,m_2})\}, \tag{61}$$

where c^0 and c^1 are two real constants given in Appendix A.3. Defining

$$A_{n_2,m_2} \equiv L_1(g^{(0,0)}_{n_2+1,m_2} - g^{(0,0)}_{n_2,m_2}) - \frac{c^0}{2}(|g^{(1,0)}_{n_2+1,m_2}|^2 + |g^{(1,0)}_{n_2,m_2}|^2)$$
$$- \frac{c^1}{2}(g^{(1,0)}_{n_2,m_2}\bar g^{(1,0)}_{n_2+1,m_2} + \bar g^{(1,0)}_{n_2,m_2} g^{(1,0)}_{n_2+1,m_2}), \tag{62}$$

Eq. (61) reads:

$$A_{n_2+1,m_2} - A_{n_2,m_2} = 0. \tag{63}$$

Thus $A_{n_2,m_2} = C(m_2)$, where $C(m_2)$ is an arbitrary function of m_2. Using the fact that $g^{(0,0)}_{n_2,m_2}$ is a slowly varying function in n_2 we have

$$L_1(g^{(0,0)}_{n_2+1,m_2} - g^{(0,0)}_{n_2-1,m_2}) = c^0(|g^{(1,0)}_{n_2+1,m_2}|^2 + |g^{(1,0)}_{n_2,m_2}|^2)$$
$$+ c^1(g^{(1,0)}_{n_2,m_2}\bar g^{(1,0)}_{n_2+1,m_2} + \bar g^{(1,0)}_{n_2,m_2} g^{(1,0)}_{n_2+1,m_2}) + C(m_2). \tag{64}$$

Equation (64) written in terms of the derivatives reads:

$$hL_1\frac{\partial g^{(0,0)}(x_2,t_2)}{\partial x_2} = (c^0 + c^1)|g^{(1,0)}_{n_2,m_2}|^2 + \frac{C(m_2)}{2}. \tag{65}$$

If we transform the derivatives of Eq. (65) into differences (using again Eq. (18), and recalling that $x_2 = hn_2$), we have

$$L_1(g^{(0,0)}_{n_2+1,m_2} - g^{(0,0)}_{n_2-1,m_2}) = 2(c^0 + c^1)|g^{(1,0)}_{n_2,m_2}|^2 + C(m_2), \tag{66}$$

an equation simpler than Eq. (64). This difference is due to the fact that Eq. (65) is obtained using the Leibniz's rule and an integration, while in the case of Eq. (64) the Leibniz's rule is not applicable as we deal with differences.

Finally, for $s = 1$, we get a system of two equations in the two unknowns, $g^{(1,2)}_{n_2,m_2}$ and $h^{(1,2)}_{n_2,m_2}$, which is compatible and not–secular only if

$$iB_1(g^{(1,0)}_{n_2,m_2+1} - g^{(1,0)}_{n_2,m_2-1}) + B_2L_1^2(g^{(1,0)}_{n_2+1,m_2} + g^{(1,0)}_{n_2-1,m_2} - 2g^{(1,0)}_{n_2,m_2})$$
$$+ B_3|g^{(1,0)}_{n_2,m_2}|^2 g^{(1,0)}_{n_2,m_2} + \{B_4(|g^{(1,0)}_{n_2+1,m_2}|^2 + |g^{(1,0)}_{n_2,m_2}|^2) + B_5(g^{(1,0)}_{n_2,m_2}\bar g^{(1,0)}_{n_2+1,m_2}$$
$$+ \bar g^{(1,0)}_{n_2,m_2} g^{(1,0)}_{n_2+1,m_2}) + B_6C(m_2)\}g^{(1,0)}_{n_2,m_2} = 0. \tag{67}$$

Here the coefficients $B_i (i = 1, \ldots, 6)$ are real and given in Appendix A.3. This is a NLSE on the lattice. At difference from the standard discrete–time NLS equation presented by

Ablowitz and Ladik [1], this is completely local but not integrable [28, 39]. In the development of $x_{n,m}$ and $y_{n,m}$, $g^{(1,0)}_{n_2,m_2}$ is the main term which multiplies ϵ^0 and $E_{n,m}$. If we require that $g^{(s,l)}_{n_2,m_2}$ and $h^{(s,l)}_{n_2,m_2}$ are localized with respect to n_2, we have to set $C(m_2) = 0$ and Eq. (67) becomes

$$\begin{aligned} & iB_1(g^{(1,0)}_{n_2,m_2+1} - g^{(1,0)}_{n_2,m_2-1}) + B_2L_1^2(g^{(1,0)}_{n_2+1,m_2} + g^{(1,0)}_{n_2-1,m_2} - 2g^{(1,0)}_{n_2,m_2}) + B_3|g^{(1,0)}_{n_2,m_2}|^2 g^{(1,0)}_{n_2,m_2} \\ & \quad + \{B_4(|g^{(1,0)}_{n_2+1,m_2}|^2 + |g^{(1,0)}_{n_2,m_2}|^2) + B_5(g^{(1,0)}_{n_2,m_2}\bar{g}^{(1,0)}_{n_2+1,m_2} \\ & \quad + \bar{g}^{(1,0)}_{n_2,m_2} g^{(1,0)}_{n_2+1,m_2})\} g^{(1,0)}_{n_2,m_2} = 0. \end{aligned} \tag{68}$$

5. Continuum Limit of the Discrete NLS

Equation (68) is obtained from Eqs. (40) and (41) by discretizing the continuous time variable. This discretization was necessary to be able to solve the $l = 1, s = 1$ system which otherwise would have been an unsolvable linear differential difference wave equation. By discretizing we get a discrete wave equation whose general solution is given by an arbitrary function of a discrete variable.

It is interesting to perform the limit when the discrete time m_1 is transformed into a continuous t–variable. To do so, we take the limit when τ goes to zero and m tends to ∞ in such a way that the product $\tau m = t$ is finite. So Eq. (68) becomes the integrable NLSE

$$iA_1\frac{\partial g^{(1,0)}(z_2,t_2)}{\partial t_2} + A_2\frac{\partial^2 g^{(1,0)}(z_2,t_2)}{\partial {z_2}^2} + [A_3|g^{(1,0)}(z_2,t_2)|^2 + A_4C(t_2)]g^{(1,0)}(z_2,t_2) = 0, \tag{69}$$

where $t_2 = \lim_{\tau\to 0}\lim_{m\to\infty}\tau m_2$ and $z_2 = \frac{1}{N}(n_1 - \frac{\mathrm{d}\Omega}{\mathrm{d}k}t_1)$ is a new continuous variable. The coefficients $A_i (i = 1, \ldots, 4)$ in this limit are finite and real, and are given by

$$\begin{aligned} A_1 &= \lim_{\tau\to 0} 2\tau B_1 = -\Omega\frac{(M_1+M_2)(k_1+k_2) - 2M_1M_2\Omega^2}{k_1+k_2-M_2\Omega^2}, \\ A_2 &= \lim_{\tau\to 0} B_2 = \frac{[(M_1+M_2)(k_1+k_2) - 2M_1M_2\Omega^2)](\Omega_{,k})^2 - M_1M_2(\Omega_{,k})^2 - k_1k_2\cos k}{k_1+k_2-M_2\Omega^2}, \\ A_3 &= \lim_{\tau\to 0}(B_3 + 2B_4 + 2B_5) = \lim_{\tau\to 0}(B_3 + 2(c_0+c_1)B_6) \\ &= -2\beta_1^2(\bar{R}-1)\left\{(R-1)|R-1|^2\frac{2k_2(1-\cos k) - (M_1+M_2)\Omega^2}{D}\right. \\ &\quad \left. + \frac{2(R-1)}{k_1+k_2}|1-R|^2\right\} + 2\beta_2^2(1-\bar{R}e^{ik})\left\{-(1-Re^{-ik})\right. \\ &\quad \left. \times |1-Re^{-ik}|^2\frac{2k_1(1-\cos k) - (M_1+M_2)\Omega^2}{D} + \frac{2(Re^{-ik}-1)}{k_1+k_2}|1-Re^{-ik}|^2\right\} \\ &\quad + 2\beta_1\beta_2(\bar{R}-1)\left\{(\bar{R}-1)(1-Re^{-ik})^2\frac{(M_2+M_1e^{2ik})\Omega^2}{D} + \frac{2(R-1)}{k_1+k_2}|1-Re^{-ik}|^2\right\} \end{aligned}$$

$$+ 2\beta_1\beta_2(1-\bar{R}e^{ik})\left\{(R-1)^2(1-\bar{R}e^{ik})\frac{(M_2+M_1e^{-2ik})\Omega^2}{D}\right.$$

$$\left. + \frac{2(1-Re^{-ik})}{k_1+k_2}|1-R|^2\right\} + 2gA_4,$$

$$A_4 = \lim_{\tau\to 0} B_6 = \frac{k_1\beta_2|1-Re^{-ik}|^2 + k_2\beta_1|1-R|^2}{k_1+k_2},$$

where

$$g = \lim_{\tau\to 0}(c_1+c_2)\frac{2\beta_2k_1|1-Re^{-ik}|^2 + 2\beta_1k_2|1-R|^2}{(M_1+M_2)(k_1+k_2)(\Omega_{,k})^2 - k_1k_2},$$

$$D = [k_1+k_2-M_1\Omega^2][k_1+k_2-M_2\Omega^2] - (k_1^2+k_2^2+2k_1k_2\cos 2k), \tag{70}$$

and

$$R = \lim_{\tau\to 0} r = \frac{k_1+k_2-M_1\Omega^2}{k_1+k_2e^{-ik}}.$$

$\Omega(k) = \lim_{\tau\to 0}\frac{\omega(k)}{\tau}$ gives back the continuous dispersion relation [9].

6. Conclusions

In this work, introducing the concepts necessary for applying the perturbative multiscale method to discrete equations we have obtained a rescaled discrete equation. We have applied this technique to a biatomic chain model. In this way we have shown that we can perform in a coherent way a multiscale expansion on the lattice. If we want to remain on the lattice and want to avoid nonlocality then we need to restrict ourselves to slow–varying functions. This restriction on the class of function implies that some of the properties of the starting system will be lost. Among them by sure that of the integrability, which is strictly related to the analytic properties of the solutions.

We have found that $g^{(1,0)}$ (the slowly varying coefficient of the first harmonic) satisfies a totally discrete local version of the discrete NLSE. One interesting feature of our discrete NLSE is that, when we perform the continuous limit in the time variable, the spatial variable becomes continuous, and we get the continuous integrable NLSE (69) as in the work by Campa *et al.* [9].

A. Appendix

A.1. $g^{(1,1)}_{n_2,m_2}$ *and* $h^{(1,1)}_{n_2,m_2}$

Let us consider the expansion of the equations of motion with $l = s = 1$. In this case we get a system of two equations in two unknowns, $g^{(1,1)}_{n_1,m_1,m_2}$ and $h^{(1,1)}_{n_1,m_1,m_2}$, that is compatible only if

$$[(k_1+k_2)(m_1+m_2) + 4m_1m_2(\cos\omega - 1)]\sin(\omega)L_2(g^{(1,0)}_{n_1,m_1+1,m_2}$$

$$- g^{(1,0)}_{n_1,m_1-1,m_2}) + k_1k_2\sin kL_1(g^{(1,0)}_{n_1+1,m_1,m_2} - g^{(1,0)}_{n_1-1,m_1,m_2}) = 0. \tag{71}$$

It is convenient to choose

$$L_1 = S\sin(\omega)[(k_1+k_2)(m_1+m_2)+4m_1m_2(\cos\omega-1)] \tag{72}$$

and

$$L_2 = Sk_1k_2\sin k, \tag{73}$$

where S is a real number such that $L_1(L_2)$ is an integer number. In terms of L_1 and L_2 the dispersion relation becomes $\omega_{,k} = \frac{L_2}{L_1}$. With this choice of L_1 and L_2, and assuming that $g^{(1,0)}_{n_1,m_1,m_2} = g^{(1,0)}_{n_2,m_2}$, with $n_2 \equiv n_1 - m_1$, we find that Eq. (71) is satisfied. Thus the system of equations we are studying is compatible, and leads us to the Eq. (57).

A.2. *The discrete NLSE*

In this Appendix, we show the steps necessary to find the discrete NLSE (68). First, we take the equations of motion, and select the harmonic $s = 1$ with $l = 2$. In this way we get a system of two equations in the two unknowns $g^{(1,2)}_{n_2,m_2}$ and $h^{(1,2)}_{n_2,m_2}$, which is compatible only if the nonhomogeneous first order difference equation

$$\begin{aligned}&[(k_1+k_2)(m_1+m_2)+4m_1m_2(\cos\omega-1)]\sin(\omega)L_2(g^{(1,1)}_{n_1,m_1+1,m_2}\\&\quad - g^{(1,1)}_{n_1,m_1-1,m_2}) + k_1k_2\sin kL_1(g^{(1,1)}_{n_1+1,m_1,m_2} - g^{(1,1)}_{n_1-1,m_1,m_2})\\&= F(g^{(0,0)}_{n_2+1,m_2}, g^{(0,0)}_{n_2-1,m_2}, g^{(1,0)}_{n_2,m_2}),\end{aligned} \tag{74}$$

is satisfied. Here $F \equiv F(g^{(0,0)}_{n_2\pm1,m_2}, g^{(1,0)}_{n_2,m_2})$ is a given nonhomogeneous term. As the l.h.s. of this equation is the same as that of Eq. (71) (but with $g^{(1,1)}_{n_2\pm1,m_2}$ replaced by $g^{(1,0)}_{n_2\pm1,m_2}$), the terms depending on $g^{(1,0)}$ contained in F lead to secular terms for the unknown $g^{(1,1)}$. To avoid secular terms, we must set $F = 0$ and Eq. (74) gives $g^{(1,1)}_{n_1,m_1,m_2} = g^{(1,1)}_{n_2,m_2}$.

If we substitute $g^{(0,0)}$ given by Eq. (64) into $F = 0$, then this condition will give Eq. (68) written in terms of $g^{(1,0)}$.

A.3. *Constants*

We give here the expressions of the coefficients appearing in Eqs. (59), (60), (64) and (68):

(1) Eqs. (59) and (60).

$$\begin{aligned}K_1 \equiv \{&\beta_1(r-1)^2[k_1+k_2e^{ik} - r(k_1+k_2e^{-2ik})]\\&- \beta_2(1-re^{-ik})^2[k_1+k_2e^{ik} - r(k_1e^{2ik}+k_2)]\}/\{rD\},\end{aligned} \tag{75}$$

$$\begin{aligned}K_2 \equiv \{&\beta_1(r-1)^2[k_1+k_2e^{2ik} - r(k_1+k_2e^{-ik})]\\&- \beta_2(1-re^{-ik})^2[k_1+k_2e^{2ik} - r(k_1e^{2ik}+k_2e^{ik})]\}/\{D\},\end{aligned} \tag{76}$$

where

$$\begin{aligned}D = &[2m_1(\cos 2\omega - 1)+k_1+k_2][2m_2(\cos 2\omega-1)+k_1+k_2]\\&- (k_1^2+k_2^2+2k_1k_2\cos 2k).\end{aligned} \tag{77}$$

(2) Eq. (64):

$$c^0 \equiv \frac{-2k_2[\beta_2|1-re^{-ik}|^2+\beta_1|1-r|^2]}{(m_1+m_2)(k_1+k_2)(\omega_{,k})^2-k_1k_2},$$

$$c^1 \equiv \frac{2\beta_2(k_1+k_2)|1-re^{-ik}|^2}{(m_1+m_2)(k_1+k_2)(\omega_{,k})^2-k_1k_2}.$$

(3) Eq. (68):

$$B_1 = -\sin(\omega)\frac{(m_1+m_2)(k_1+k_2)+4m_1m_2(\cos\omega-1)}{2m_2(\cos\omega-1)+k_1+k_2},$$

$$B_2 = \frac{\begin{array}{c}[(m_1+m_2)(k_1+k_2)+4m_1m_2(\cos\omega-1)]\\ \cos(\omega)(\omega_{,k})^2-m_1m_2 sin^2(\omega)(\omega_{,k})^2-k_1k_2\cos k\end{array}}{k_1+k_2+2m_2(\cos\omega-1)},$$

$$\begin{aligned}
B_3 = &-2\beta_1^2(\bar{r}-1)\left\{(r-1)|r-1|^2\frac{2k_2(1-\cos k)+2(m_1+m_2)(\cos\omega-1)}{D}\right.\\
&+\left.\frac{2(r-1)}{k_1+k_2}|1-r|^2\right\}\\
&+2\beta_2^2(1-\bar{r}e^{ik})\left\{-(1-re^{-ik})|1-re^{-ik}|^2\frac{2k_1(1-\cos k)+2(m_1+m_2)(\cos\omega-1)}{D}\right.\\
&+\left.\frac{2(re^{-ik}-1)}{k_1+k_2}|1-re^{-ik}|^2\right\}\\
&+2\beta_1\beta_2(\bar{r}-1)\left\{(\bar{r}-1)(1-re^{-ik})^2\frac{-2e^{2ik}m_1(\cos\omega-1)-2m_2(\cos\omega-1)}{D}\right.\\
&+\left.\frac{2(r-1)}{k_1+k_2}|1-re^{-ik}|^2\right\}\\
&+2\beta_1\beta_2(1-\bar{r}e^{ik})\left\{(r-1)^2(1-\bar{r}e^{ik})\frac{-2e^{-2ik}m_1(\cos\omega-1)-2m_2(\cos\omega-1)}{D}\right.\\
&+\left.\frac{2(1-re^{-ik})}{k_1+k_2}|1-r|^2\right\},
\end{aligned}$$

$$B_4 = c^0B_6,$$

$$B_5 = c^1B_6,$$

$$B_6 = \frac{k_1\beta_2|1-re^{-ik}|^2+k_2\beta_1|1-r|^2}{k_1+k_2}.$$

References

[1] M. J. Ablowitz and J. F. Ladik, Nonlinear differential-difference equations, *J. Math. Phys.* **16** (1975) 598–603.

[2] M. Agrotis, S. Lafortune and P. G. Kevrekidis, On a discrete version of the Korteweg–de Vries equation, *Discr. Cont. Dyn. Syst.* **suppl.** (2005) 22–29.

[3] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders College Publ., 1976).

[4] G. Assanto, A. Fratalocchi and M. Peccianti, Spatial solitons in nematic liquid crystals: from bulk to discrete, *Optics Express* **15** (2007) 5248–5259.

[5] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientist and Engineers I* (Springer Verlag, Berlin, 1999).

[6] B. H. Bransden and C. J. Joachain, *Physics of Atoms and Molecules* (Longman, London, 1983).

[7] F. Calogero and W. Eckhaus, Nonlinear evolution equations, rescalings, model PDEs and their integrability: I, *Inv. Prob.* **3** (1987) 229–262.

[8] F. Calogero and W. Eckhaus, Nonlinear evolution equations, rescalings, model PDEs and their integrability: II, *Inv. Prob.* **4** (1988) 11–33.

[9] A. Campa, A. Giansanti, A. Tenenbaum, D. Levi and O. Ragnisco, Quasisolitons on a diatomic chain at room temperature, *Phys. Rev. B* **48** (1993) 10168–10182.

[10] O. A. Chubykalo, V. V. Konotop and L. Vázquez, Small-amplitude solitary waves on a lattice subject to nonvanishing boundary conditions, *Phys. Rev. B* **47** (1993) 7971–7977.

[11] P. C. Dash and K. Patnaik, Nonlinear wave in a diatomic Toda lattice, *Phys. Rev. A* (3) **23** (1981) 959–969.

[12] P. C. Dash and K. Patnaik, Solitons in nonlinear diatomic lattices, *Prog. Theor. Phys.* **65** (1981) 1526–1541.

[13] A. Di Bucchianico and D. Loeb, Umbral Calculus, *Electron. J. Combin.* **DS3** (2000).

[14] E. Fermi, J. Pasta and S. Ulam, Los Alamos Rpt. LA-1940 (1955); *Collected Papers of Enrico Fermi* (Univ. of Chicago Press, Chicago) Vol. II (1965), p. 978.

[15] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, Method for solving the Korteweg–deVries equation, *Phys. Rev. Lett.* **19** (1967) 1095–1097.

[16] B. I. Henry and J. Oitmaa, Dynamics of a nonlinear chain, *Aust. J. Phys.* **36** (1983) 339–356.

[17] R. Hernández Heredero, D. Levi, M. Petrera and C. Scimiterna, Multiscale expansion of the lattice potential KdV equation on functions of an infinite slow-varyness order, *J. Phys. A: Math. Theor.* **40** (2007) F831–F840.

[18] C. Jordan, *Calculus of Finite Differences* (Röttig and Romwalter, Sopron, 1939).

[19] J. Kevorkian and J. D. Cole, *Multiple Scale and Singular Perturbation Methods*, Applied Mathematical Sciences 114 (Springer–Verlag, New York, 1996).

[20] D. J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves, *Phil. Mag.* **39** (1895) 422–443.

[21] R. A. Kraenkel, M. A. Manna and J. G. Pereira, The Korteweg–de Vries hierarchy and long water waves, *J. Math. Phys.* **36** (1995) 307–320.

[22] J. Leon and M. Manna, Multiscale analysis of discrete nonlinear evolution equations, *J. Phys. A* **32** (1999) 2845–2869.

[23] D. Levi and R. Hernández Heredero, Multiscale analysis of discrete nonlinear evolution equations: the reduction of the dNLS, *J. Nonl. Math. Phys.* **12** (2005) 440–448.

[24] D. Levi and M. Petrera, Discrete reductive perturbation technique, *J. Math. Phys.* **47** (2006) 043509.

[25] D. Levi and P. Tempesta, Multiscale analysis of dynamical systems on the lattice, *J. Math. Analysis Appl.*, in press.

[26] F. Mokross and H. Büttner, Comments on the diatomic Toda lattice, *Phys. Rev. A* **3** (1981) 2826–2828.

[27] O. H. Olsen, M. R. Samuelsen, S. B. Petersen and L. Norskov, Amide-I excitations in molecular-mechanics models of a helix structures, *Phys. Rev. A* **39** (1989) 3130–3134.

[28] A. Ramani, private communication.

[29] S. Roman, *The Umbral Calculus* (Academic Press, New York, 1984).

[30] G. C. Rota, *Finite Operator Calculus* (Academic Press, New York, 1975).

[31] J. S. Russel, Report on waves, Report on the fourteenth meeting of the British Association Adv. Sci. (1845) 311–390.

[32] S. W. Schoombie, A discrete multiple scales analysis of a discrete version of the Korteweg–de Vries Equation, *J. Comp. Phys.* **101** (1992) 55–70.

[33] A. Scott, *Nonlinear Science* (OUP, Oxford, 1999).
[34] A. Shelkan, V. Hizhnyakov and M. Koplov, Self-consistent potential of intrinsic localized modes: Application to diatomic chain, *Phys. Rev. B* **75** (2007) 134304.
[35] T. Taniuti, Reductive perturbation method and far fields of wave equations, *Suppl. Progr. Theor. Phys.* **55** (1974) 1–35.
[36] T. Taniuti and C. C. Wei, Reductive perturbation method in nonlinear wave propagation. I, *J. Phys. Soc. Jap.* **24** (1968) 941–946.
[37] M. Toda, *Theory of Nonlinear Lattices*, Springer Series in Solid State Sciences 20 (Springer–Verlag, Berlin, 1989).
[38] M. Toda, *Theory of Nonlinear Waves and solitons* (Kluwer, Dordrecht, 1989).
[39] C. Viallet, private communication.
[40] N. Yajima and J. Satsuma, Soliton solutions in a diatomic lattice system, *Prog. Theor. Phys.* **62** (1979) 370–378.
[41] R. Yamilov, Symmetries as integrability criteria for differential difference equations, *J. Phys. A* **39** (2006) R541–R623.
[42] N. J. Zabusky and M. D. Kruskal, Interaction of solitons in a collisionless plasma and the recurrence of initial states, *Phys. Rev. Lett.* **16** (1965) 240–243.
[43] N. J. Zabusky, Fermi–Pasta–Ulam, solitons and the fabric of nonlinear and computational science: history, synergetics, and visiometrics, *CHAOS* **15** (2005) 015102.